Computer Robust Output Feedback Stabilizing Control of Decentralized Singularly-Perturbed Systems

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ABSTRACT

In this paper, robust output feedback stabilizing controllers are developed for computer control of decentralized singularly-perturbed systems without state estimation devices. The developed techniques will be suitable for decentralized large-scale and high-dimension systems with strong potential for practical applications, such as aircraft, power distribution and communication networks. Moreover, the controllers are expected to satisfy following criteria: (1) computer control, (2) decentralized control, (3) robust control, (4) stabilizing control, (5) reliable control, (6) reduced-order control, and (7) output feedback control. The analog to digital transformation technique is used to convert continuous-time state models to discrete-time domain for digital control use. Singular perturbation methods are applied to reduce the order of the system. The concept of Riccati equation approach will be applied to stabilize the systems. Moreover, the robustness test will be fulfilled to understand the robustness bound of the system, and reliable control investigation will be performed to show the reliability of the system.

Keywords: robust, stability, output feedback, decentralized, singularly-perturbed, order-reduction, reliability.

分散式殊異擾動系統之電腦強健式輸出回授穩定性控制

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摘　　要

在這篇文章中，發展出一種可由電腦執行的強健式輸出回授穩定性控制技術，且不需要有觀察器的輔助。系統狀態將由輸出信號獲得。此控制技術將可應用於大型分散式殊異擾動系統。且有助於實際上的應用，如電力系統，網路系統和通訊系統。

控制器的設計包含以下之特性：(1)電腦控制、(2)分散式控制、(3)強健控制、(4)穩定性控制、(5)可靠性控制、(6)減階控制、(7)輸出回授控制。類比/數位轉換技術被用來轉換連續時間的狀態模組至離散時間，以達成數位控制的目的。奇異擾動法則被用來作減階的技術，穩定性控制可以用最佳化控制法來獲得。且強健控制測試也將被實行，藉以了解強健控制之範圍值。再者，也亦將進行系統之可靠性控制分析。

關鍵字: 強健、穩定性、輸出回授、分散式、殊異擾動、減階、穩定性

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I. INTRODUCTION

The computer control becomes a major technique in today's large-scale and high-dimension systems. These systems include electric power systems, nuclear reactors, aerospace systems, computer networks, communication networks, and the petroleum industry. Such systems consist of a number of independent subsystems that serve particular functions, share resources, and are governed by a set of interrelated goals and constraints. Each of the subsystems is responsible for the operation of a specific task of the overall system or does full-function control. This situation is often referred to as decentralization. Operations of decentralized control are performed on the subsystem level that make it attractive in cases where the classical stabilization and estimation techniques are either impossible or impractical to apply due to the excessive dimensionality of the system. Therefore, decentralized control is a major control scheme for dealing with large-scale systems. Decentralized control is also important for the following reasons: (1) The construction of a central control unit and the transmission lines among the various subsystems are costly, particularly for geographically divided systems such as electrical power networks. (2) The transmission of a signal among the various controlled units can delay and distort the transfer of information. (3) The instrumentation necessary for carrying out a decentralized control is simple. (4) As a result of the technological growth of computer control, the cost of local control units is constantly decreasing. (5) Decentralized control enables the local control units to be completely independent and thereby increases the robustness of the system. It is not surprising that in past decade many researchers have directed their attention to various problems that relate to such systems. Such systems are also called interconnected systems. Moreover, they are usually endowed with a complex interconnecting structure and are frequently high-dimension. Consequently, researchers have tried to simplify these interconnecting high-dimension systems.

The decentralized robust controllers in large-scale systems have been discussed and designed by many different approaches in many papers [1-3]. Usually, there are three kinds of controllers: (1) state feedback controllers, (2) output feedback controllers [1, 4] and (3) observer-based controllers [2, 5]. Many papers concentrate on output feedback controllers and observer-based controllers, because state variables of systems are not always available for measurement. Output feedback controllers and observer-based controllers can perform sub-optimal control, because in output feedback controllers, the obtained optimal gains are for output feedback not state feedback; in observer-based controllers, state variables estimated are not exactly correct values and observers are the necessary equipment in this control scheme. After these concerns, output feedback control becomes a control method that many engineers and researchers are interested in, because it is less expensive and more reliable [6].

The decentralized singularly-perturbed system may also be stabilized by optimal output feedback control. Savkin and Petersen [1] investigated the stability of continuous time full-order linear systems via decentralized output feedback control. They use only the input-output information and applying the linear quadratic regulator [LQR]. Yan, Wang, Lu and Zhang [7] investigated the stability of decentralized output feedback for class of nonlinear systems. They developed a controller with holographic structure for the stability of large-scale decentralized systems with similarity. But, these methods are not suitable for the system that this paper is looking into.

In systems, stabilization is the most important part when we design controllers. Therefore, controllers not only have to achieve the performances we expect but they also must be able to stabilize systems. There are many approaches used by researchers to investigate the stability and the stabilization of decentralized singularly-perturbed systems. Among all system performance requirements, robust stability is a paramount condition for designs of system control. Numerous approaches have been proposed in the literature. Especially, in [8], the system concerned is also a singularly-perturbed system. However, in this paper, the system is not only a singularly-perturbed system but also a decentralized control system and a computer controlled system.

One of the major approaches to robust stabilization is by the Riccati equation. This approach can also achieve the optimal control [9]. However, it is well known that if attention is restricted to linear time-invariant controllers, then the use of decentralized control may lead to decentralized fixed modes. The presence of such fixed modes may prevent a given system from being stabilized via linear time-invariant decentralized control. After all, this is a particular situation. We can assume the models are stabilizable and detectable to prevent fix modes. However, in this research, our motivation for introducing such an output feedback controller is not to overcome the problem of fixed modes but rather to give a controller that is stabilizing with respect to the quadratic performance index.

The controllers not only stabilize the system but also achieve desire performance is always the major task of the designs. Therefore, the pole placement of the systems is also needed to concern when finding the stabilizing gain. The LQR regulator would be a good approach, because the technique is well known and familiar by all engineers.

Often in control design it is necessary to construct estimates of state variables which are not available by direct measurement. If a system is linear, its state vector can be approximately reconstructed by building an observer which is a linear system driven by the available outputs and inputs of the original system. Lunberger [10] first proposed an "observer" and introduced the idea of a reduced-order observer to estimate those states of a system that are not accessible by direct measurement. The other type of observer is the same order as the process under observation. They are called full-order observers. In this paper, we propose reduced-order output feedback control which doesn't need to construct observers if the state variables are not available for measurement.

Robust control is one major concern for the designs in large-scale systems. Here, the control scheme of decentralized singularly-perturbed systems will be investigated for robust control. The state model will be always an inaccurate representation of the actual physical system because of parameter changes, unmodeled dynamics, unmodeled time delays, changes in equilibrium point, sensor noise, and unpredicted disturbance inputs. A robust control system exhibits the desired performance despite the presence of significant uncertainty. In this paper, the unmodeled dynamics will be the major uncertainty considered, because decentralized singularly-perturbed systems are reduced to lower-order systems by neglecting the fast state variables that only affect the system responses in the very initial time period. Therefore, in this research, robust control is defined as that the desired performances is still existing after applying the decentralized reduced-order output feedback controllers in singularly-perturbed full-order systems. The bound of robust control is expected to be found in the investigation.

The reliability of the control systems is also a relevant evaluation criterion. Conventional feedback control designs for a multi-input-multi-output plant may result in unsatisfactory control system performance or even instability. In the event of controller outages, it may be possible to control the plant using only the surviving inputs and outputs. A control system designed to tolerate failures of controllers, while retaining desired control system properties, will be called a "reliable" control system. In the decentralized control, the reliability goal is the stabilization of the system by a controller in each control channel, such that the system can tolerate control channel failures [11]. Siljack [12] presents the approach of designing a separate stabilizing controller for each control channel, and gives "connective stability" conditions under which some or all of these controllers together also stabilize the system.

Due to increasing complexity of modern technological process, reliability of control has become an essential requirement in the design of large-scale systems. In a decentralized high-dimension system, if a local controller breaks down, it is entirely feasible that the whole system may do the same. Replacement of a faulty controller by a standby, or disconnection of the corresponding subsystem for the purpose of preventing the system breakdown may be either impossible or undesirable due to the design constraints. A reliability goal for a decentralized system is the stabilization of the plant by a controller in each control channel [11, 12].

In decentralized control systems, basically there are two types of control schemes. The control scheme of the type-one, the controller of each channel only controls partial overall state variables [12]. The control scheme of the type-two, the controller of each channel can control overall state variables [11]. The second type of decentralized control scheme is considered as a potential reliable control scheme, because of the structure. In the type-two system, if the controller of each channel is able to stabilize the system, the system possesses reliable control properties. Due to the way of designs, the decentralized control scheme that has been developed in this research belongs to the type-two system.

In output feedback controller designs, singular perturbation methods, computer control schemes, and decentralization have never been involved together such as this research. In today's system control, computers are widely used in every sophisticated system; therefore, computer control implementation should be absolutely and highly concerned in the design procedure. Using digital computers to implement controllers has substantial advantages. Many of the difficulties with analog implementation can be avoided. For example, there is no problem with accuracy or drift of the components. It is very easy to have sophisticated calculations in the control law, and it is easy to include logic and nonlinear function. Tables can be used to store data in order to accumulate knowledge about the properties of the system. It is also possible to have effective user interfaces [13,14]. Using reduced-order control can helps to simplify the system analysis, improves the manufacturing process, and minimizes the cost.

II. PROBLEM STATEMENT

The mathematical model of the system is shown as:

$$\begin{cases} \dot{x} = A_{00}x + \sum_{i=1}^{m} A_{0i} z_i & \text{(1a)} \\ \varepsilon\dot{z}_i = A_{i0}x + A_{ii}z_i + B_i u_i & \text{(1b)} \end{cases}$$

$$y_i = C_i z_i \qquad \text{(1c)}$$

where i=1~m, which can be also shown as

$$\begin{cases} \dot{x} = A_{00}x + A_{01}z_1 + A_{02}z_2 + A_{03}z_3 + \ldots\ldots A_{0m}z_m \\ \varepsilon\dot{z}_1 = A_{10}x + A_{11}z_1 + B_1u_1 \\ \varepsilon\dot{z}_2 = A_{20}x + A_{22}z_2 + B_2u_2 \\ \varepsilon\dot{z}_3 = A_{30}x + A_{33}z_3 + B_3u_3 \\ \vdots \\ \varepsilon\dot{z}_m = A_{m0}x + A_{mm}z_m + B_mu_m \end{cases} \qquad \text{(1d)}$$

$$\begin{cases} y_1 = C_1z_1 \\ y_2 = C_2z_2 \\ y_3 = C_3z_3 \\ \vdots \\ y_m = C_mz_m \end{cases} \qquad \text{(1e)}$$

The system is a linear time-invariant decentralized singularly-perturbed system which has n-order and m independent inputs or m sub-systems. $x \in R^S$ and $z \in R^F$ are the slow and the fast state variables respectively; each sub-system z_i has its own order. $u_i \in R^{n_{1i}}$ and $y_i \in R^{n_{2i}}$ are the input vector of the i-th subsystem and the output vector of the i-th subsystem respectively. A_{00}, A_{0i}, A_{i0}, A_{ii} and C_i are constant matrices with appropriate dimensions with i=1~m. A_{11} ~ A_{mm} are nonsingular matrices.

The goal of this research is to develop computer control of robust, output feedback, reduced-order controllers for stabilization of linear singularly-perturbed systems via decentralized control. The system control scheme is shown as Fig. 1.

The computer control schemes will be employed to implement the proposed output feedback controllers. Singular perturbation methods will be used to reduce the order of the model. Hence, the proposed techniques will be suitable for large-scale and high-dimension systems with strong potential for practical applications, such as power distribution and communication networks.

This research provides a new technique to control decentralized singularly perturbed systems. It also helps to simplify the system analysis, improves the manufacturing process, minimizes the cost, and stabilizes the systems.

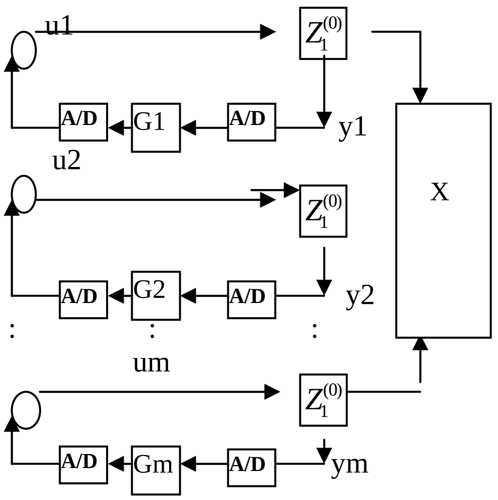

Fig.1. The computer output feedback control of decentralized singularly perturbed systems.

The value of the small parameter ε will be the key point when we discuss the robustness of the designed controllers in uncertain systems. When we assume $\varepsilon \cong 0$, that means we neglect the fast state variables that belong to the sub-systems in this case. This is also how we can improve the manufacturing process and simplify the system analysis. A standard LQ criterion will be given as a condition for the optimal control design. This will help us to minimize the cost and stabilize the system [1].

Computer control implementation will be fulfilled, and appropriate A/D devices and sampling rates will be used to ensure the robustness of proposed designs.

III. PRELIMINARIES

3.1 Singular Perturbation Methods

Using the state-variables representation, a linear time-invariant system can be represented as

$$\begin{cases} \dot{x} = A_{11}x + A_{12}z + B_1 u, \quad x(t=0) = x(0) & \text{(2a)} \\ \varepsilon\dot{z} = A_{21}x + A_{22}z + B_2 u, \quad z(t=0) = z(0) & \text{(2b)} \end{cases}$$

where x and z are m- and n- dimensional slow state and fast state of the system. u is an r-dimensional control vector. The matrices A_{ij} and B_i are of appropriate dimensions. The scalar, positive parameter ε represents all small parameters to be ignored. The system (2) is in the singularly perturbed form in the sense that by making ε = 0 in the system. the degenerate system becomes

$$\begin{cases} \dot{x}^{(0)} = A_{11}x^{(0)} + A_{12}z^{(0)} + B_1 u, \ x^{(0)}(t=0) = x(0) & \text{(3a)} \\ 0 = A_{21}x^{(0)} + A_{22}z^{(0)} + B_2 u, \ z^{(0)}(t=0) \neq z(0) & \text{(3b)} \end{cases}$$

or

$$\begin{cases} \dot{x}^{(0)} = (A_{11} - A_{12}A_{22}^{-1}A_{21})x^{(0)} + (B_1 - A_{12}A_{22}^{-1}B_2)u & \text{(4a)} \\ z^{(0)}(t) = -A_{22}^{-1}A_{21}x^{(0)} - A_{22}^{-1}B_2 u & \text{(4b)} \end{cases}$$

Therefore, the full, high-order, or perturbed system (2) becomes the degenerate, low-order, or unperturbed system [15,16].

3.2 Analog to Digital Transformation

In order to obtain the digital control laws and fit computer control schemes, continuous-time state models needed to be transformed to discrete-time state models [13].

A given state model

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & \text{(5a)} \\ y(t) = Cx(t) & \text{(5b)} \end{cases}$$

The complete response of the model is

$$x(t) = \phi(t - t_k)x(t_k) + \int_k^t \phi(t-\tau)Bu(\tau)d\tau \qquad (6)$$

where ϕ = transition matrix.

But, $u(\tau) = u(t_k) =$ constant; since $t_k < \tau < t$ and zero-order-hold sampling has been used. So,

$$x(t) = \phi(t - t_k)x(t_k) + \int_k^t \phi(t-\tau)B(t_k)d\tau \qquad (7)$$

$$= \phi(t-t_k)x(t_k) + \int_k^t \phi(t-\tau)d\tau Bu(t_k) \quad (8)$$

let $\lambda = t - \tau$ (9)

therefore; $d\lambda = -d\tau$

when $\tau = t_k \rightarrow \lambda = t - t_k$

$\tau = t \rightarrow \lambda = 0$

$$x(t) = \phi(t-t_k)x(t_k) + \int_{\lambda=t-t_k}^{\lambda=0} \phi(\lambda)(-d\lambda)Bu(t_k) \quad (10)$$

$$= \phi(t-t_k)x(t_k) + \int_0^{t-t_k} \phi(\lambda)(d\lambda)Bu(t_k) \quad (11)$$

set $t = t_{k+1}$, where $t_{k+1} - t_k = h$ =constant sampling period

$$x(t_{k+1}) = \phi(t_{k+1}-t_k)x(t_k) + \int_0^{t_{k+1}-t_k} \phi(\lambda)(d\lambda)Bu(t_k) \quad (12)$$

$$= \phi(h)x(t_k) + \int_0^h \phi(\lambda)d\lambda Bu(t_k) \quad (13)$$

Now, we set $t_k = kh$ where h is the fixed sampling interval, and k is an integer index that represents the number of sampling periods. Hence, we can represent $t_{k+1} = (k+1)h$. Now, the discrete-time model of the continuous-time system is shown as

$$x((k+1)h) = \phi(h)x(kh) + \int_0^h \phi(\lambda)d\lambda Bu(kh) \quad (14)$$

In general; $x((k+1)h) = A_d x(kh) + B_d u(kh)$ (15)

where $A_d = \phi(h) = \Phi$ (16)

$$B_d = \int_0^h \phi(\lambda)d\lambda B = \Gamma \quad (17)$$

Also, in the output equation, C is unchanged; therefore, $y(t) = Cx(t)$ yields $y(kh) = Cx(kh)$.

3.3 Stabilization and LQR Optimal Control

If the discrete-time system is

$$\begin{cases} x(k+1) = \Phi x(k) + \Gamma u(k) & (18) \\ y(k) = Cx(k) & (19) \end{cases}$$

The closed-loop system is

$$x(k+1) = (\Phi - \Gamma K)x(k) = A_C x(k) \quad (20)$$

and the LQ performance index is

$$J = \frac{1}{2}\sum_{k=0}^{N}(x^T(k)Qx(k) + u^T(k)Ru(k)) \quad (21)$$

Then, the optimal control

$$u^{optimal}(k) = -K(k)x(k) \quad (22)$$

where $K = (\Gamma^T P \Gamma + R)^{-1}\Gamma^T P \Phi$ (23)

and, the P is the solution of the Riccati equation

$$P = \Phi^T\{P - P\Gamma[\Gamma^T P \Gamma + R]^{-1}\Gamma^T P\}\Phi + Q \quad (24)$$

where P is a constant matrix.

This optimal control feedback gain is also the gain that can stabilize the system [1].

IV. MAIN RESULTS

Fig. 2 shows the approach to control the system. Fig. 3 shows the idea of the order reduction in the step one of Fig 2. The system is a two-time scale system; therefore, there are fast state variables and slow state variables. The slow state variables belong to the main controlled system, and the fast state variables belong to the control subsystems. The responses of the fast state variables will die out very fast so that we can just ignore the fast state variables and base the design on a reduced-order system.

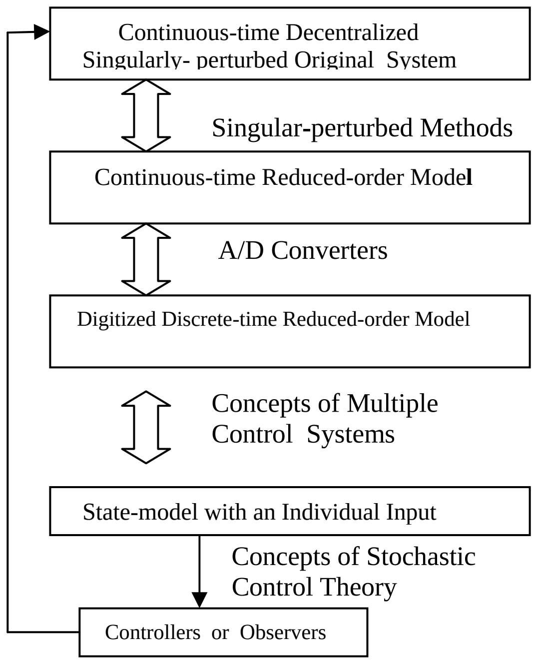

Fig.2. The flow-diagram of the approach.

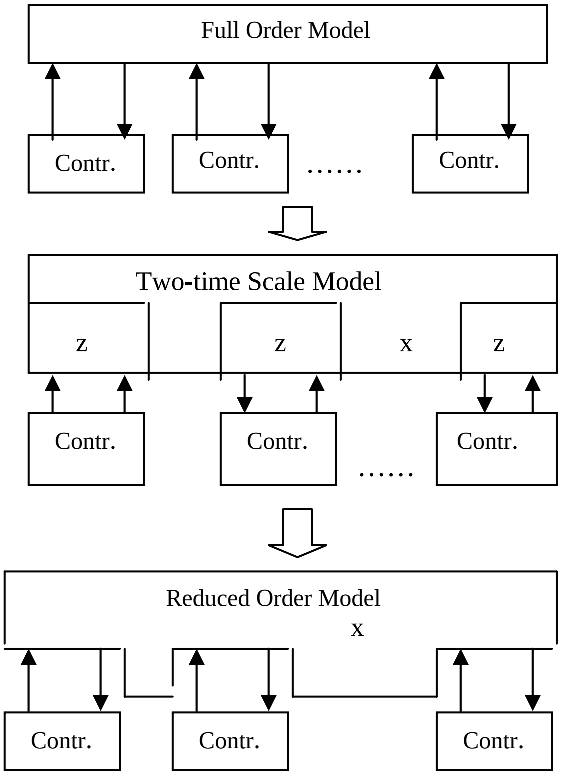

Fig.3. The idea of order reduction.

Because the overall system is a computer controlled system, the responses of the computer based subsystems are a lot faster than the main plant, and the responses of the fast state variables will die out pretty fast in the very initial time period. Due to this phenomenon, the fast state variables can be ignored to simplify the system; the order of the overall system can be reduced. This also rises the idea that the state model of the overall system can be approximated. This kind of system status can refer to Quasi-steady state, and the parameter ε can be set to be zero.

In this controller design, singular perturbation methods are applied to reduce the order of the system. The system model is given by Eq(1). The system is in the singularly perturbed form in the sense that ε =0. So, the sub-station station variables, z_1, z_2, z_3 ...... have reached quasi-steady state. Hence, the system order is reduced to the order of the main station which is equal to the dimension of the slow state variable x.

Therefore, from Eq(1), we can re-build the state model as

$$\dot{x} = A_{00}x + A_{01}z_1 + A_{02}z_2 + A_{03}z_3 + ...A_{0m}z_m \qquad (25a)$$

$$\begin{bmatrix} A_{11} & 0 & 0 & .. & .. & .. & .. & .. & 0 & 0 \\ 0 & A_{22} & 0 & .. & .. & .. & .. & .. & .. & 0 \\ 0 & 0 & A_{33} & .. & .. & .. & .. & .. & .. & : \\ : & .. & .. & A_{44} & .. & .. & .. & .. & .. & : \\ : & .. & .. & .. & A_{55} & .. & .. & .. & .. & : \\ : & : & : & : & : & : & : & : & : & : \\ : & : & : & .. & : & : & : & : & : & : \\ : & : & : & : & : & : & : & : & : & : \\ 0 & : & : & : & : & : & : & : & : & 0 \\ 0 & 0 & .. & .. & .. & .. & .. & .. & 0 & A_{mm} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ : \\ : \\ : \\ : \\ : \\ : \\ z_m \end{bmatrix} = \begin{bmatrix} \varepsilon \dot{z}_1 - B_1 u_1 - A_{10} x \\ \varepsilon \dot{z}_2 - B_2 u_2 - A_{20} x \\ \varepsilon \dot{z}_3 - B_3 u_3 - A_{30} x \\ : \\ : \\ : \\ : \\ : \\ : \\ \varepsilon \dot{z}_m - B_m u_m - A_{m0} x \end{bmatrix} \qquad (25b)$$

The system is in the singularly perturbed form in the sense that ε =0. So, the sub-station station variables, z_1, z_2, z_3 ...... have reached quasi-steady state. Hence, the system order is reduced to the order of the main station which is equal to the dimension of the slow state variable x. From Eq(25b), it can be shown as

$$\begin{bmatrix} A_{11} & 0 & 0 & .. & .. & .. & .. & .. & 0 & 0 \\ 0 & A_{22} & 0 & .. & .. & .. & .. & .. & .. & 0 \\ 0 & 0 & A_{33} & .. & .. & .. & .. & .. & .. & : \\ : & .. & .. & A_{44} & .. & .. & .. & .. & .. & : \\ : & .. & .. & .. & A_{55} & .. & .. & .. & .. & : \\ : & : & : & : & : & : & : & : & : & : \\ : & : & : & .. & : & : & : & : & : & : \\ : & : & : & : & : & : & : & : & : & : \\ 0 & : & : & : & : & : & : & : & : & 0 \\ 0 & 0 & .. & .. & .. & .. & .. & .. & 0 & A_{mm} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ : \\ : \\ : \\ : \\ : \\ : \\ z_m \end{bmatrix} = \begin{bmatrix} - B_1 u_1 - A_{10} x \\ - B_2 u_2 - A_{20} x \\ - B_3 u_3 - A_{30} x \\ : \\ : \\ : \\ : \\ : \\ : \\ - B_m u_m - A_{m0} x \end{bmatrix} \qquad (26)$$

Next, in order to solve $\begin{bmatrix} z_1 \\ z_2 \\ : \\ z_m \end{bmatrix}$, by inspection from Eq(25a), we can see the diagonal matrices of A's are square; therefore, we obtain:

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ : \\ : \\ : \\ : \\ : \\ : \\ z_m \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & 0 & .. & .. & .. & .. & .. & 0 & 0 \\ 0 & A_{22} & 0 & .. & .. & .. & .. & .. & .. & 0 \\ 0 & 0 & A_{33} & .. & .. & .. & .. & .. & .. & : \\ : & .. & .. & A_{44} & .. & .. & .. & .. & .. & : \\ : & .. & .. & .. & A_{55} & .. & .. & .. & .. & : \\ : & : & : & : & : & : & : & : & : & : \\ : & : & : & : & : & : & : & : & : & : \\ : & : & : & : & : & : & : & : & : & : \\ 0 & : & : & : & : & : & : & : & : & 0 \\ 0 & 0 & .. & .. & .. & .. & .. & .. & 0 & A_{mm} \end{bmatrix}^{-1} \begin{bmatrix} -B_1u_1 - A_{10}x \\ -B_2u_2 - A_{20}x \\ -B_3u_3 - A_{30}x \\ : \\ : \\ : \\ : \\ : \\ : \\ -B_mu_m - A_{m0}x \end{bmatrix} \quad (27)$$

We expand the two matrices on the right. We can have Eq(28) shown below. Equation (28) shows that all sub-systems z_i can be represented by the slow state vector x.

From (1d), it can be shown as

$$\begin{cases} z_1 = A_{11}^{-1}(-B_1u_1 - A_{10}x) = -A_{11}^{-1}B_1u_1 - A_{11}^{-1}A_{10}x \\ z_2 = A_{22}^{-1}(-B_2u_2 - A_{20}x) = -A_{22}^{-1}B_2u_2 - A_{22}^{-1}A_{20}x \\ z_3 = A_{33}^{-1}(-B_3u_3 - A_{30}x) = -A_{33}^{-1}B_3u_3 - A_{33}^{-1}A_{30}x \\ \qquad : \\ z_m = A_{mm}^{-1}(-B_mu_m - A_{m0}x) = -A_{mm}^{-1}B_mu_m - A_{mm}^{-1}A_{m0}x \end{cases} \quad (28)$$

So, $z_i = -A_{ii}^{-1}B_iu_i - A_{ii}^{-1}A_{i0}x$ (29)

where i=1~m; $A_{11} \sim A_{mm}$ are nonsingular matrices.

Next, recall the state equation of the slow state variable in Eq(1a) which is

$$\dot{x} = A_{00}x + \sum_{i=1}^{m} A_{0i}z_i \quad (30)$$

We can obtain new representations for the equation of slow state variables by using Eq(29).

$$\dot{x} = A_{00}x + \sum_{i=1}^{m} A_{0i}z_i \quad (31a)$$

$$= A_{00}x + \{\sum_{i=1}^{m}[A_{0i}\ (-A_{ii}^{-1}B_iu_i - A_{ii}^{-1}A_{i0}x)]\} \quad (31b)$$

$$= [A_{00} - \sum_{i=1}^{m} A_{0i}A_{ii}^{-1}A_{i0}]x + [\sum_{i=1}^{m} - A_{0i}A_{ii}^{-1}B_iu_i] \quad (31c)$$

$$= [A_{00} - \sum_{i=1}^{m} A_{0i}A_{ii}^{-1}A_{i0}]x + \left[-A_{01}A_{11}^{-1}B_1u_1 \quad -A_{02}A_{22}^{-1}B_2u_2 \quad .... \quad -A_{0m}A_{mm}^{-1}B_mu_m\right]\begin{bmatrix} u_1 \\ u_2 \\ : \\ u_m \end{bmatrix} \quad (31d)$$

Now, an n-order multi-input decentralized singularly-perturbed system is reduced into a S=(n-F)-order multi-input time-invariant system.

So, from Eq(31), we can write the reduced-order model as

$$\begin{cases} \dot{x}_r = A_rx_r + B_ru & (32a) \\ y_i = C_iz_i = C_{ri}x_r + D_{ri}u_i & (32b) \end{cases}$$

where x_r indicates that the model has been processed with order-reduction

$z_i = -A_{ii}^{-1}B_iu_i - A_{ii}^{-1}A_{i0}x_r$; $C_{ri} = -C_iA_{ii}^{-1}A_{i0}$

$D_{ri} = -C_iA_{ii}^{-1}B_i$

$A_r = [A_{00} - \sum_{i=1}^{m} A_{0i}A_{ii}^{-1}A_{i0}]$

$B_r = \left[-A_{01}A_{11}^{-1}B_1 \quad -A_{02}A_{22}^{-1}B_2 \quad ... \quad -A_{0m}A_{mm}^{-1}B_m\right]$

$u = \begin{bmatrix} u_1 \\ u_2 \\ : \\ u_3 \end{bmatrix}$

Next step, from the model Eq(32), the model can be digitized as [15].

$$x_r((k+1)h) = \Phi x_r(kh) + \Gamma u(kh) \quad (33)$$

where $\Phi = \phi(h) = e^{A_rh}$

$\Gamma = \int_0^h \phi(\lambda)d\lambda B$

h : sampling period.

Equation (33) can be also shown as

$$x_r(k+1) = \Phi x_r(k) + \Gamma_1 u_1(k) + ...\Gamma_m u_m(k) \quad (34)$$

where Γ_1 is the first column of Γ and Γ_2 is the second column of Γ and so on.

Now, if the main station is controlled from the subsystem one. The Eq(34) can be revised as

$$x_r(k+1) = \Phi_N x_r(k) + \Gamma_1 u_1(k) \quad (35a)$$

where $\Phi_N = (\Phi + \Gamma_2 K_2 + \Gamma_3 K_3 + ...\Gamma_m K_m)$
and $K_2 \sim K_m$ are existing assumed state feedback gains.

$$y_1(k) = C_{r1} x_r(k) + D_{r1} u_1(k) \quad (35b)$$

If the state variables are not available for measurement, we may use system output that is always available. In the subsystem one, the output feedback can be assumed as

$$u_1(k) = G_1 y_1(k) \quad (36)$$

where G_1 is the output feedback gain of the subsystem one.

Optimization will be performed based on the condition of Eq(36), This means we are looking for the optimal gain G^* not the optimal input $u^*(k)$; therefore, this control can be also called the sub-optimal control.

From Eq(35b) and Eq(36),

$$u_1(k) = G_1[C_{r1} x_r(k) + D_{r1} u_1(k)] \quad (37)$$

so, $[I - G_1 D_{r1}] u_1(k) = G_1 C_{r1} x_r(k)$ (38)

where I is an identity matrix.

then $u_1(k) = [I - G_1 D_{r1}]^{-1} G_1 C_{r1} x_r(k)$

or, $u_1(k) = P_1 x_r(k)$ (39)

where $P_1 = [I - G_1 D_{r1}]^{-1} G_1 C_{r1}$

The state model of the close-loop output feed back control from the subsystem one can be written as

$$x_r(k+1) = A_C x_r(k) \quad (40)$$

where $A_C = \Phi_N + \Gamma_1 P_1$

The LQ performance index of each subsystem is

$$J_i = \frac{1}{2} \sum_{k=0}^{N-1} (w^T(k) Q_i w(k) + u_i^T(k) R_i u_i(k)) \quad (41)$$

where Q_i: the weighting matrix with p. s. d. for each sub-system and $Q_i = \begin{bmatrix} Q_{11}^i & 0 \\ 0 & Q_{22}^i \end{bmatrix}$.

R_i : the weighting matrix with p. d. for each sub-system.

$w = \begin{bmatrix} x_r \\ z \end{bmatrix}$; x_r is a slow state vector and z is a fast state vector.

Also, the LQ performance index of unknown slow state x_r can be obtained as [17].

$$J_i = \frac{1}{2} \sum_{k=0}^{N-1} (x_r^T(k) Q_{ri} x_r(k) + u_i^T(k) R_i u_i(k)) \quad (42)$$

Therefore, for the subsystem one, the performance index of the discrete-time reduced-order model Eq(35) can be shown as

$$J_1 = \frac{1}{2} \sum_{k=0}^{N-1} (x_r^T(k) Q_{r1} x_r(k) + u_1^T(k) R_1 u_1(k)) \quad (43)$$

Now, from the preliminaries, this LQR optimal control state feedback gain also stabilizes the system. Therefore, we know if we have control from the subsystem one, based on Eq(35) and Eq(43), a stabilizing state feedback type gain can be found as

$$u^S(k) = -K_1(k) x_r(k) \quad (44)$$

where $K_1 = (\Gamma_1^T L \Gamma_1 + R_1)^{-1} \Gamma_1^T L \Phi_N$ (45)

and, the L is the solution of the Riccati equation

$$L = \Phi_N^T \{L - L\Gamma_1[\Gamma_1^T L \Gamma_1 + R_1]^{-1} \Gamma_1^T L\} \Phi_N + Q_{r1} \quad (46)$$

where L is a constant matrix.

Now, let Eq(39) equals to Eq(44), the output feedback gain of the subsystem one, G_1, can be found by the following process.

$$P_1 x_r(k) = -K_1(k) x_r(k) \quad (47a)$$

$$P_1 = [I - G_1 D_{r1}]^{-1} G_1 C_{r1} = -K_1 \quad (47b)$$

$$G_1 C_{r1} = -[I - G_1 D_{r1}] K_1 \quad (47c)$$

$$G_1 C_{r1} = -K_1 + G_1 D_{r1} K_1 \quad (47d)$$

$$G_1 [C_{r1} - D_{r1} K_1] = -K_1 \quad (47e)$$

In Eq(47e), the matrix $[C_{r1} - D_{r1}K_1]$ maybe a non-square matrix. In order to move the matrix $[C_{r1} - D_{r1}K_1]$ to the right side of the equal sign, the pseudo-inverse matrix can be applied in here. By defining a pseudo-inverse matrix, $T = [C_{r1} - D_{r1}K_1]$. The output feedback gain of the subsystem one is found as

$$G_1 = -K_1 T^R \tag{48}$$

where T^R is pseudo-right inverse which is $T^T(TT^T)^{-1}$.
K_1 can be any matrix that satisfies Eq (45) and Eq (46).

Next, using the matrix G_1 to obtain $P_1 = [I - G_1 D_{r1}]^{-1} G_1 C_{r1}$ and the close-loop output feedback control is implemented by P_1 as Eq(39). In Eq(47a), the reduced-order state $x_r(k)$ is cancelled in both sides; therefore, the state of this system would not be concerned in this control scheme. It achieves the goal of no state estimation needed.

Since the close-loop system is stable, the output feedback gain G_1 is a robust stabilizing gain. The system can be stabilized by this output feedback gain. The robust stabilizing output feedback gain of the subsystem two to the subsystem m can just follow the same procedure as the subsystem one.

Due to the control scheme is designed based on the reduced order model, the stability of the reduced order model is assured by applying the Riccati Equation shown in the previous procedure. For the stability of the full order system, a robustness test is needed for understanding the robustness bound of the system. Furthermore, the stability of the overall system can be confirmed. The robustness test procedure is shown in the illustration. Table 1 contains the result of the robustness test of the illustration problem.

V. RELIABLE CONTROL INVESTIGARION

In decentralized systems, reliable control is obtained, if every controller of every subsystem is able to stabilize the system. Even if one of the controllers fails, the rest of the controllers can still control the overall system. Therefore, the system can avoid breakdown problem. Siljack presents the approach of designing a separate stabilizing controller for each control channel. In such a structure, the system is called a multiple control system. It has been established that the system possessing this type of structure has build-in reliability properties [12]. Figure 4 shows the structure of a multiple control system.

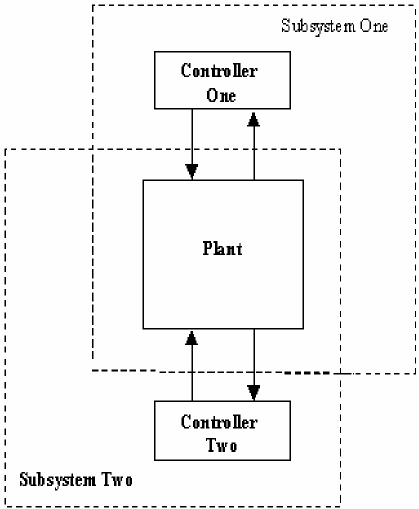

Fig.4. A multiple control scheme.

On the contrary, the other type of decentralized control scheme that has a state Eq(49) is shown in Fig. 5. [12,17,18]. In this type of decentralized systems, the controller of each channel does not have ability to control overall state variables. In Eq(49), the term $u_i(k)$ influences $\sum_{\substack{j=1 \\ j\neq i}}^{N} \Phi_{ij} x_j(k)$, but it is unable to control the poles of $x_j(k)$. $u_i(k)$ can only control $x_i(k)$.

$$x_i(k+1) = \Phi_i x_i(k) + \Gamma_i u_i(k) + \sum_{\substack{j=1 \\ j\neq i}}^{N} \Phi_{ij} x_j(k) \tag{49}$$

where i is the number of the controlled unites and j is the number of the subsystems, and, $i = j$.

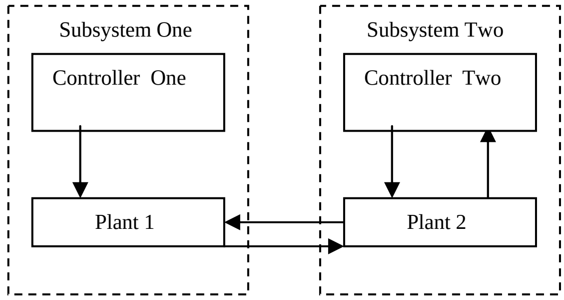

Fig.5. A decentralized control system that the controllers only control partial overall state variables.

From Fig. 5, the overall system breaks down due to a disconnection or a failure of the subsystem is shown in Fig. 6.

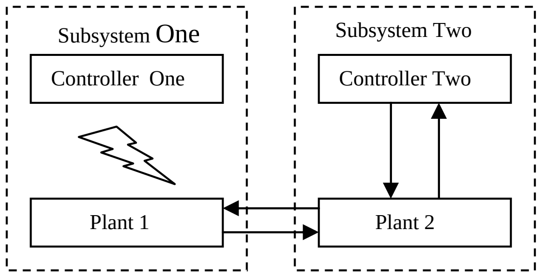

Fig.6. A controller failure.

From Fig. 6, the controller of the subsystem one fails. The plant one is unable to be controlled by the controller one. Although the controller two is still functioning, it only can control the plant two that is only part of the overall system.

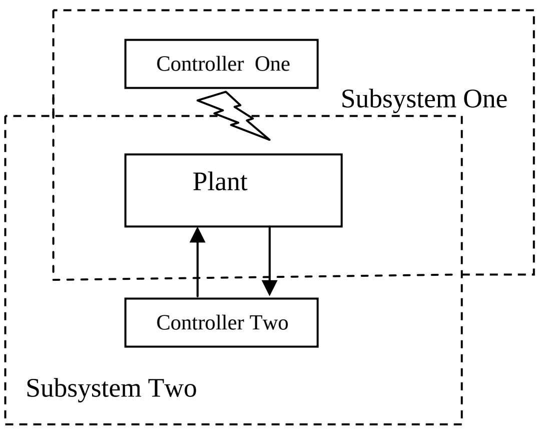

Figure 7. A multiple control system with a failure controller.

On the other hand, if the multiple control system has same failure that is shown in Fig. 7, the controller two can still control overall system regardless of the malfunction of the controller one.

From the concepts of multiple control systems, to design a reliable decentralized control system can be based on the structure of decentralized control systems.

Now let us look at the structure of a decentralized singularly-perturbed system shown in Figure 1 and the state equation of the subsystem Eq(50) developed in the previous process.

The reason why Eq(50) is discussed for reliable control is to see whether the output feedback controller G_1 can control overall state variables.

$$x_r(k+1) = \Phi_N x_r(k) + \Gamma_1 u_1(k) \tag{50a}$$

$$= \Phi_N x_r(k) + \Gamma_1 P_1 x_r(k) \tag{50b}$$

$$= [\Phi_N + \Gamma_1 P_1] x_r(k) \tag{50c}$$

where $\Phi_N = (\Phi + \Gamma_2 K_2 + \Gamma_3 K_3 + ...\Gamma_m K_m)$

and $K_2 \sim K_m$ are existing assumed state feedback gains.

$P_1 = [I - G_1 D_{r1}]^{-1} G_1 C_{r1}$

After choosing an appropriate value for G_1, the matrix $\Gamma_1 P_1$ will be absorbed to the matrix Φ_N; therefore, a $n \times n$ matrix $[\Phi_N + \Gamma_1 P_1]$ that contains the existing poles of the system is obtained. In other words, the output feedback controller G_1 can control the overall state variables.

From Eq(50) and Fig. 1, the input of the subsystem one

$$u_1(k) = G_1 y_1(k) = P_1 x_r(k) \tag{51}$$

It obviously shows that the input $u_1(k)$ can control overall state variables. It is likewise in the rest of the inputs and in the full order system. In addition, the stabilizing output feedback controller has been developed by the Riccati equation approach in Eq(48). Therefore, these two concepts conclude that the controller of each channel can stabilize the overall state variables. This is same as the idea of a multiple control system proposed by Siljack [14].

VI. ILLUSTRATION

A system is a fifth-order system with three first order subsystems, three inputs and three outputs. The state model is shown as

$$\begin{bmatrix} \dot{x} \\ \varepsilon\dot{z}_1 \\ \varepsilon\dot{z}_2 \\ \varepsilon\dot{z}_3 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0.1 & -0.2 & 0.1 \\ 0 & -1 & 0.1 & 0.3 & -0.2 \\ 0.4 & -0.3 & -0.5 & 0 & 0 \\ -0.4 & 0.4 & 0 & -0.45 & 0 \\ 0.35 & 0.3 & 0 & 0 & -0.4 \end{bmatrix} \begin{bmatrix} x \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.4 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & 0.6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad (52a)$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \quad (52b)$$

where x is a slow state vector that is second-order.

z_1, z_2, z_3 are all fast state vectors and first-order individually.

Therefore, when we set $\varepsilon = 0$, the system can be reduced to a second order system such as

$$\dot{x}_r = \begin{bmatrix} -0.1545 & -0.163 \\ -0.363 & -0.943 \end{bmatrix} x_r + \begin{bmatrix} 0.08 & -0.222 & 0.15 \\ 0.08 & 0.333 & -0.3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad (53)$$

Next, we digitize this reduced-order model to discrete-time domain with the sampling period 0.1.

$$x_r(k+1) = \begin{bmatrix} 0.985 & -0.0154 \\ -0.0344 & 0.9103 \end{bmatrix} x_r(k) + \begin{bmatrix} 0.0079 & -0.0223 & 0.0151 \\ 0.0075 & 0.0322 & -0.0289 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \\ u_3(k) \end{bmatrix} \quad (54)$$

Now, if we want to have the optimal control in the subsystem one, by assuming the existing

$P_2 = [1 \quad 1]$ and $P_3 = [5 \quad 5]$, we can rewrite the model as

$$x_r(k+1) = \begin{bmatrix} 1.0382 & 0.0378 \\ -0.1467 & 0.7980 \end{bmatrix} x_r(k) + \begin{bmatrix} 0.0079 \\ 0.0075 \end{bmatrix} u_1(k) \quad (55)$$

If the performance index of the slow state vector from the subsystem one is

$$J_1 = \frac{1}{2} \sum_{k=0}^{N-1} (x_r^T(k) Q_1 x_r(k) + u_1^T(k) R_1 u_1(k)) \quad (56)$$

where $Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $R_1 = 1$

The robust stabilizing controller of the subsystem one

$$u_1^s = [-3.2814 \quad -0.5737]\, x_r(k) \quad (57)$$

with $L = \begin{bmatrix} 364.8580 & 61.0517 \\ 61.0517 & 13.3957 \end{bmatrix}$

Next, from Eq(12) $C_{ri} = -C_i A_{ii}^{-1} A_{i0}$ and $D_{ri} = -C_i A_{ii}^{-1} B_i$. After digitization, the values and the sizes of C_{r1} and D_{r1} are unchanged. Therefore, we can obtain $C_{r1} = [0.8 \quad -0.6]$ and $D_{r1} = 0.8$. Moreover, the output feedback gain of the subsystem one is also obtained by Eq(48). $G_1 = -KT^R$; Therefore, $G_1 = 1.48$. Also, $P_1 = [I - G_1 D_{r1}]^{-1} G_1 C_{r1} = [-6.4. \quad 4.8]$. To ensure that G_1 is a stabilizing gain, we can use the value of P_1 in Eq(39) and apply into Eq(55). Then, the close-loop system model would be like Eq(40) that $A_C = \begin{bmatrix} 0.9876 & 0.0757 \\ -0.1947 & 0.8340 \end{bmatrix}$. The poles of the original open-loop system that has $\Phi_N = \begin{bmatrix} 1.0382 & 0.0378 \\ -0.0344 & 0.7980 \end{bmatrix}$ with eigenvalues 1.0123 and 0.8239 has one pole outside the unit circle and unstable. After using the output feedback gain, G_1, the poles of the close-loop system become $0.9108 \pm i0.094$ which are inside the unit circle and stable. Therefore, it shows that G_1 is a robust stabilizing gain. The control scheme of the subsystem one can be shown as Fig. 8. Moreover, the output feedback gains of the subsystem two and the subsystem three can be found by the same procedure as the subsystem one.

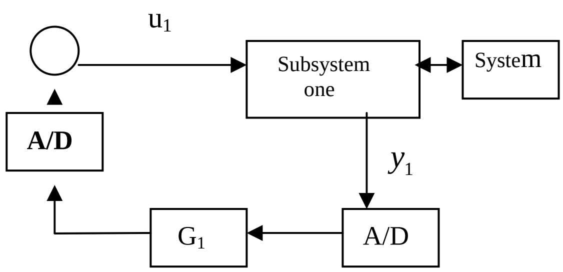

Fig.8. The computer robust output feedback stabilizing controller of the subsystem one.

6.1 The robust control test

Using the same technique and same conditions in the original full-order model with $\varepsilon = 0.001$. The discrete-time model would be like

$$w(k+1)=\begin{bmatrix} 0.9842 & -0.0151 & 0.0002 & -0.0004 & 0.0003 \\ -0.0334 & 0.9102 & 0.0002 & 0.0004 & -0.0005 \\ 0.8072 & -0.559 & 0 & -0.0007 & 0.0005 \\ -0.9042 & 0.8239 & 0 & 0.0001 & -0.0006 \\ 0.837 & 0.6713 & 0.0003 & 0.0001 & -0.0001 \end{bmatrix} w(k)+ \begin{bmatrix} 0.0077 & 0.00218 & 0.0147 \\ 0.0074 & -0.0315 & -0.0282 \\ 0.8017 & 0.0356 & 0.0281 \\ -0.0003 & -1.1574 & -0.0373 \\ 0.012 & -0.0045 & 1.4919 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \\ u_3(k) \end{bmatrix}$$

Now, we apply the feedback gains obtained from the reduced-order model in the original full-order model. We will have the locations of the two main performing poles at $0.90 \pm i0.092$. We can see the two poles' locations are very close to the locations that we find in the reduced-order model. That shows robust control of this reduced-order output feedback controller.

Also, by assuming $y_1 = [0 \quad 0 \quad 1 \quad 1 \quad 1]\begin{bmatrix} x \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}$, we can have the system responses based on the subsystem one with h=0.1 and ε=0.001 as follows.

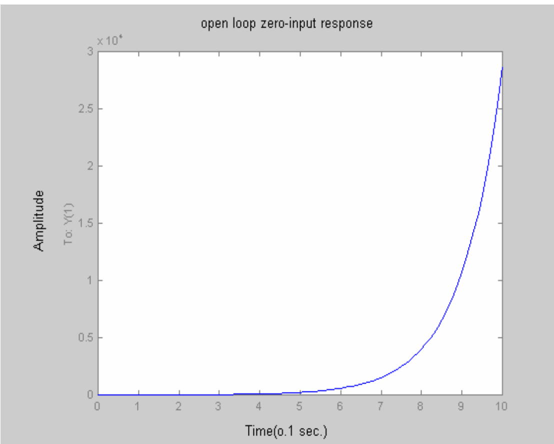

Fig.9. The open-loop zero-input response of the full order system with the slow state poles at 1.012 and 0.824 ; P_2=[1 1],P_3=[5 5] respect to subsystem one. The system is unstable.

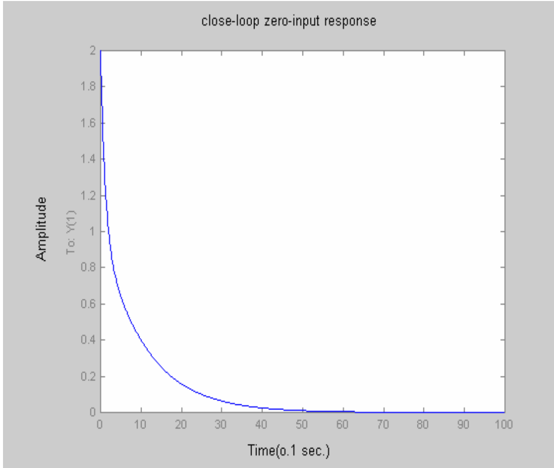

Fig.10. The close-loop zero-input response of the full-order system with the reduced-order output feedback controller shifting poles to $0.90 \pm i0.092$ respect to subsystem one. The system is stable.

We can also find out how the system tolerates system uncertainties by changing the parameter ε. Table 1 shows how the poles shift when the value of ε changes.

Table 1. The robust control test.

ε	Poles
5.0000e-004	$0.897 \pm i0.092$
0.0060	$0.899 \pm i0.0922$
0.0115	$0.902 \pm i0.0925$
0.0170	$0.907 \pm i0.0927$
0.0225	$0.909 \pm i0.0929$
0.0280	$0.91 \pm i0.093$
0.0335	$0.911 \pm i0.0931$
0.0390	$0.912 \pm i0.0932$
0.0445	$0.913 \pm i0.0933$

Every system has difference tolerance from system uncertainties. In this case, we assume the system performance allows shift of radius = 0.012 at each pole location. Then, when ε < 0.017, we can have a robust control system. The reduced-order controllers that perform inside this bound are called robust, decentralized reduced-order output feedback controllers. Moreover, the stability of the full order system that is controlled by the reduced order controllers can be also confirmed if the control is inside the robustness bound. The complete procedure achieves the design of computer robust output stabilizing control.

The same procedure can be used in the subsystem two and the subsystem three for the robust control test.

VII. CONCLUSIONS

The goal of this paper is to find the robust stabilizing gain that can be used by the decentralized output feedback controller. The computer implemented reduced-order controllers can stabilize the decentralized singularly-perturbed systems from every substation. This more reliable and cheaper output feedback control technique overcomes the disadvantages of building observers in each system. Now, with the found robust stabilizing gain of the reduce-order control, the computer control technique of decentralized singularly-perturbed systems is getting more mature on all aspects.

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