The Study of Mutually Independent Hamiltonian Paths of Crossed Cubes

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ABSTRACT

The topology of a network is usually denoted by a graph $G=(V,E)$, where nodes represent processors and edges represent links between processors. Two Hamiltonian paths $P_1 =< u=u^0,u^1,...,u^{|V|}=v >$, $P_2 =< u=w^0,w^1,...,w^{|V|}=v >$ of G are independent if all the nodes distinct (i.e., $u^i \neq w^i$ for all $0<i<|V|$) except the source node (i.e., $u=u^0=w^0$) and destination node(i.e., $v=u^{|V|}=w^{|V|}$). A set of Hamiltonian paths {$P_1,P_2,...,P_n$} in G are mutually independent if any two distinct Hamiltonian paths in the set are independent each other. A n-dimensional crossed cube CQ_n is a variant of hypercube, it has many attractive properties. In this paper, we prove that there are 4 mutually independent Hamiltonian paths in CQ_n with $n \geq 5$.

Keywords: crossed cubes, Hamiltonian path, independent path, interconnection networks, hypercubes

交錯式立方體中相互獨立的漢米爾頓路徑研究

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摘 要

一個相互連結網路(Interconnection Network)之拓樸結構通常是以圖形 $G=(V,E)$ 表示，圖形中的點 V 代表網路上之處理器，線 E 則為處理器與處理器間的連線。兩條漢米爾頓路徑(Hamiltonian Paths)，$P_1 =< u=u^0,u^1,...,u^{|V|}=v >$ 及 $P_2 =< u=w^0,w^1,...,w^{|V|}=v >$ 是獨立的，亦即除了起始結點與目的結點相同外，其餘結點均兩兩不相同；一組漢米爾頓路徑 $P_1,P_2,...,P_n$ 若兩兩均為獨立的則稱為相互獨立的漢米爾頓路徑(Mutually Independent Hamiltonian Paths)。一個 n 維交錯式立方體(Crossed Cubes)是超立方體(Hypercubes)的一種變形。本論文我們證明當 $n \geq 5$ 時，交錯式立方體結構 CQ_n 內存在 4 條相互獨立的漢米爾頓路徑。

關鍵詞：交錯式立方體，漢米爾頓路徑，獨立路徑，連結網路，超立方體

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Ⅰ. INTRODUCTION

The topology of a network is usually denoted by a graph $G=(V,E)$, where nodes represent processors and edges represent links between processors. The crossed cube CQ_n has been studied extensively in recent years because the crossed cube is a variant of hypercube and it has many attractive properties [1-12] such as diameter, wide diameter and fault diameter, conditional fault diameter are approximately half of those of the hypercube [1,2]. The power of the crossed cube simulates trees, paths and cycles are stronger than those of the hypercube [1,11-12]. Furthermore, the connectivity, diagnosability, bisection width are equal to those of the hypercube. In [13], the mutually independent paths in star networks and hypercubes have been proven that there are node degree minus one. The mutually independent Hamiltonian paths of CQ_n is studied in this work.

Two Hamiltonian paths $P_1=<u=u^0,u^1,...,u^{|V|}=v>$, $P_2=<u=w^0,w^1,...,w^{|V|}=v>$ of G are independent which is all the nodes distinct (i.e., $u^i \neq w^i$ *for all* $0<i<|V|$) except the source node (i.e., $u=u^0=w^0$) and destination node (i.e., $v=u^{|V|}=w^{|V|}$). A set of Hamiltonian paths $\{P_1,P_2,...,P_n\}$ in G are *mutually independent* if any two distinct Hamiltonian paths in the set are independent each other. A graph is Hamiltonian connected if there is a Hamiltonian path joining each pair of nodes. A cycle is a path (except the first node is the same as the last node) containing at least three nodes. A cycle of G is a Hamiltonian cycle if it contains all vertices. A graph is Hamiltonian if it has a Hamiltonian cycle. In [14], Huang et al. studied the fault-tolerant Hamiltonian path embedding in crossed cubes, and it showed that the faulty CQ_n is still Hamiltonian with up to $n-2$ faults. There are many researchers investigating the Hamiltonian-connectivity of various interconnection networks [15-18], the crossed cube is Hamiltonian connected [16]. In [19,20], the frequency channels assignment of Hamiltonian path is better than that of tree for broadcasting over wireless communication. For broadcasting a message, it can be divided into sub-messages transmitted via mutually independent Hamiltonian paths. Thus, mutually independent Hamiltonian paths are considered in this paper. How to construct mutually independent Hamiltonian paths in CQ_n is essential to this work. Let u and v be arbitrarily distinct nodes of CQ_n. Thus, we show that there are four mutually independent Hamiltonian paths from u to v in CQ_n with $n\geq 5$.

The rest of this paper is organized as follows: Section 2 summarizes some known results on crossed cubes and introduces notation used in this paper. In Section 3, we prove our result. In the final section, we give our conclusion.

Ⅱ. PRELIMINARIES AND NOTATION

Let $G=(V,E)$ represent a graph, where V represents the node set of G and E the edge set of G. Let x and y be two nodes. We use $d(x,y)$ to denote the distance between x and y in G. To define crossed cubes, we first introduce the notation of "*pair related*". The notation is $x_1x_0 \sim y_1y_0$. Let $T=\{(00,00),(01,11),(10,10),(11,01)\}$. Two binary strings $x=x_1x_0$ and $y=y_1y_0$ are *pair related* if and only if $(x,y)\in T$.

Definition 1. The n-dimensional crossed cube CQ_n recursively constructed as follows: CQ_1 is a complete graph with two nodes labeled by 0 and 1, respectively. CQ_n consists of two identical subcubes CQ_{n-1}^0 and CQ_{n-1}^1 . The node $u=0u_{n-2}...u_0\in V(CQ_{n-1}^0)$ and $v=1v_{n-2}...v_0\in V(CQ_{n-1}^1)$ is an edge in CQ_n if and only if (1) $u_{n-2}=v_{n-2}$ if n is even, and (2) $(u_{2i+1}u_{2i},v_{2i+1}v_{2i})\in T$ for $0\leq i<\lceil (n-1)/2\rceil$.

Figure 1 shows CQ_3 and CQ_4 . For $k<n$, the k-prefix of u, $p_k(u)$, is defined as $u_{n-1}u_{n-2}...u_{n-k}$. Each node in CQ_n can thus write $u=u_{n-1}...u_0=p_k(u)u_{n-k-1}...u_0$.

Let x be an l-bit string with $l\leq n$.

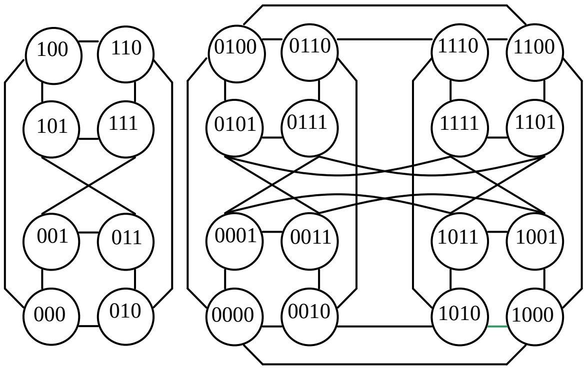

Fig.1. CQ_3 and CQ_4.

$CQ_n(x)$ denotes the subgraph of CQ_n induced by the set of nodes with prefix x. It was shown in [10] that $CQ_n(x)$ is isomorphic to $CQ_{n-|x|}$.

Let x and y be two distinct l-bit strings with $l < n$. If $CQ_n(x)$ and $CQ_n(y)$ can be joined by an edge in CQ_n, then $CQ_n(x)$ and $CQ_n(y)$ are called *adjacent* subgraphs of CQ_n. Let $CQ_n(x,y)$ denote the subgraph of CQ_n induced by $CQ_n(x) \cup CQ_n(y)$. It was proven in [4] that $CQ_n(x,y)$ is isomorphic to $CQ_{n-|x|+1}$ if $CQ_n(x)$ and $CQ_n(y)$ are adjacent subgraphs of CQ_n.

Let $P(u,v) = < u = z^0, z^1, ..., z^{m-1}, z^m = v >$ be a paths in CQ_n. Given the paths $P(u,v)$, z^{m-1} is called the *immediate predecessor* of z^m. The length of $P(u,v)$ is denoted by $l(P(u,v))$. The paths $P(u,v)$ can also be written as $< u = z^0, z^1, P(z^1,v) >$. Furthermore, we use $R(w;P)$ to denote the paths in CQ_n induced from $P(u,v)$, which is given by $R(w;P) = < w = w^0, w^1, ..., w^m >$, where $p_{n-2}(w^i) = p_{n-2}(z^i)$ for all $0 \le i \le m$. Obviously, the paths length of $l(R(w;P))$ is $l(R(w;P)) = l(P(p_{n-2}(u), p_{n-2}(v)))$. Two paths, $P(u,v) = < u, z^1, z^2, ..., z^{m-1}, z^m, v >$ and $Q(u,v) = < u, z^1, s^2, ..., s^{m-1}, z^m, v >$, are partially disjoint if $z^i \ne s^j$ for all $2 \le i \le m-1$ and $2 \le j \le m-1$.

Definition 2. Let (u,v) be an edge of CQ_n. When nodes u and v have a leftmost differing bit at position j, we say that v is the j-neighbor of u and the edge (u,v) is an edge of dimension j.

For example, let $u = 10101$. The 4-, 3-, 2-, 1-, and 0-neighbors of u are given by 01111, 11111, 10011, 10111, and 10100, respectively. The $N_i(u)$ represents the i-neighbor of u. Similarly, given $u = 10101$, the edge of dimension 4, 3, 2, 1, and 0 is represented as 01111, 11111, 10011, 10111, and 10100, respectively. The edge dimension can simplify the representation of node if the routing has a long distance.

To derive the mutually independent Hamiltonian paths in CQ_n, the following lemmas shown in [1], [12] is preliminary.

Lemma 1 [1]. *CQ_n is a pancyclic network for $n \ge 2$, i.e., CQ_n contains a cycle of length l for all $4 \le l \le 2^n$ as subgraphs.*

Lemma 2 [12]. *There are 2n! distinct Hamiltonian paths in CQ_n for all $n \geq 3$.*

Ⅲ. MUTUALLY INDEPENDENT HAMILTONIAN PATHS OF CROSSED CUBES

By Lemma 2, although there are $2n!$ distinct Hamiltonian paths in CQ_n, there are not mutually independent Hamiltonian paths. Hence, the mutually independent Hamiltonian paths in CQ_n is investigated in this work.

In order to construct independent Hamiltonian paths in CQ_n, we define the path $P_i = P_{i,S} \cup \{e_{L1}\} \cup P_{i,I} \cup \{e_{L2}\} \cup P_{i,D}$, where $\{e_{L1}\}$ and $\{e_{L2}\}$ represent the dimension $L1$ and $L2$ of edges, respectively. Paths $P_{i,S}$, $P_{i,I}$ and $P_{i,D}$ represent the *source* path, the *intermediate* path and the *destination* path, respectively. These paths are given as follows.

Lemma 3. *Let u be a source node in $CQ_4^{x_1x_0}$ and D be a set of ending nodes in $CQ_4^{y_1y_0}$. Then $CQ_4(x_1x_0)$ and $CQ_4(y_1y_0)$ are adjacent subgraphs if two binary strings $y_1y_0 = \overline{x_1}x_0$ or $y_1y_0 = x_1\overline{x_0}$. There exists four mutually independent paths in CQ_4 from u to nodes $v \in D$.*

Proof: We first define the *source* paths, there are four mutually independent paths from the source node $u \in CQ_4^{x_1x_0}$ to a set of ending nodes D as follows.

Case 1. $D \subseteq CQ_4^{\overline{x_1}x_0}$.

Four mutually independent paths are from $u = x_1x_000$ to $v = \overline{x_1}x_000$ 、 $\overline{x_1}x_001$ 、 $\overline{x_1}x_011$ and $\overline{x_1}x_010$, respectively. There are given by

$$P_{1,S} = \langle \underline{x_1x_000}, x_1x_001, x_1\overline{x_0}11, x_1\overline{x_0}01, x_1x_011, x_1x_010, x_1\overline{x_0}10, x_1\overline{x_0}00, \overline{x_1}\,\overline{x_0}00, \overline{x_1}\,\overline{x_0}01, \overline{x_1}\,\overline{x_0}11, \overline{x_1}\,\overline{x_0}10, \overline{x_1}x_010, \overline{x_1}x_000, \overline{x_1}x_001, \underline{\overline{x_1}x_011} \rangle.$$

$$P_{2,S} = \langle \underline{x_1x_000}, x_1x_010, x_1x_011, x_1x_001, x_1\overline{x_0}11, x_1\overline{x_0}10, x_1\overline{x_0}00, x_1\overline{x_0}01, \overline{x_1}\,\overline{x_0}11, \overline{x_1}\,\overline{x_0}10, \overline{x_1}\,\overline{x_0}00, \overline{x_1}\,\overline{x_0}01, \overline{x_1}x_011, \overline{x_1}x_010, \overline{x_1}x_000, \underline{\overline{x_1}x_001} \rangle.$$

$$P_{3,S} = \langle \underline{x_1x_000}, x_1\overline{x_0}00, x_1\overline{x_0}01, x_1\overline{x_0}11, x_1x_001, x_1x_011, x_1x_010, x_1\overline{x_0}10, \overline{x_1}\,\overline{x_0}10, \overline{x_1}\,\overline{x_0}00, \overline{x_1}\,\overline{x_0}01, \overline{x_1}\,\overline{x_0}11, \overline{x_1}x_001, \overline{x_1}x_011, \overline{x_1}x_010, \underline{\overline{x_1}x_000} \rangle.$$

$$P_{4,S} = \langle \underline{x_1x_000}, \overline{x_1}x_000, \overline{x_1}x_001, \overline{x_1}x_011, \overline{x_1}\,\overline{x_0}01, \overline{x_1}\,\overline{x_0}00, \overline{x_1}\,\overline{x_0}10, \overline{x_1}\,\overline{x_0}11, x_1\overline{x_0}01, x_1\overline{x_0}00, x_1\overline{x_0}10, x_1\overline{x_0}11, x_1x_001, x_1x_011, x_1x_010, \underline{\overline{x_1}x_010} \rangle.$$

Obviously, there are four mutually independent paths in the *source* paths except the source node. To simplify the problem, we can use the edges' dimension to represent the others *source* paths, from one of source node x_1x_001, x_1x_011 or x_1x_010 to a set of ending nodes $D = \{\overline{x_1}x_000, \overline{x_1}x_001, \overline{x_1}x_011, \overline{x_1}x_010\}$, we list different *source* paths from each source node in Table 1.

Case 2. $D \in CQ^{x_1\overline{x_0}}$.

Four mutually independent paths are from $u = x_1x_000$ to $v = x_1\overline{x_0}00$, $x_1\overline{x_0}01$, $x_1\overline{x_0}11$ and $x_1\overline{x_0}10$, respectively. There are given by

$$P_{1,S} = \langle \underline{x_1x_000}, x_1x_001, \overline{x_1}x_011, \overline{x_1}x_001, x_1x_011, x_1x_010, \overline{x_1}x_010, \overline{x_1}x_000, \overline{x_1}\,\overline{x_0}00, \overline{x_1}\,\overline{x_0}01, \overline{x_1}\,\overline{x_0}11, \overline{x_1}\,\overline{x_0}10, x_1\overline{x_0}10, x_1\overline{x_0}00, x_1\overline{x_0}01, \underline{x_1\overline{x_0}11} \rangle.$$

$$P_{2,S} = \langle \underline{x_1x_000}, x_1x_010, x_1x_011, x_1x_001, \overline{x_1}x_011, \overline{x_1}x_010, \overline{x_1}x_000, \overline{x_1}x_001, \overline{x_1}\,\overline{x_0}11, \overline{x_1}\,\overline{x_0}10, \overline{x_1}\,\overline{x_0}00, \overline{x_1}\,\overline{x_0}01, x_1\overline{x_0}11, x_1\overline{x_0}10, x_1\overline{x_0}00, \underline{x_1\overline{x_0}01} \rangle.$$

$$P_{3,S} = \langle \underline{x_1x_000}, x_1\overline{x_0}00, x_1\overline{x_0}01, x_1\overline{x_0}11, \overline{x_1}\,\overline{x_0}01, \overline{x_1}\,\overline{x_0}00, \overline{x_1}\,\overline{x_0}10, \overline{x_1}\,\overline{x_0}11, \overline{x_1}x_001, \overline{x_1}x_000, \overline{x_1}x_010, \overline{x_1}x_011, x_1x_001, x_1x_011, x_1x_010, \underline{x_1\overline{x_0}10} \rangle.$$

$$P_{4,S} = \langle \underline{x_1x_000}, \overline{x_1}x_000, \overline{x_1}x_001, \overline{x_1}x_011, x_1x_001, x_1x_011, x_1x_010, \overline{x_1}x_010, \overline{x_1}\,\overline{x_0}10, \overline{x_1}\,\overline{x_0}00, \overline{x_1}\,\overline{x_0}01, \overline{x_1}\,\overline{x_0}11, x_1\overline{x_0}01, x_1\overline{x_0}11, x_1\overline{x_0}10, \underline{x_1\overline{x_0}00} \rangle.$$

We also use the edges' dimension to represent the others *source* paths, from each source node x_1x_001 、 x_1x_011 or x_1x_010 to a set of ending nodes $D = \{x_1\overline{x_0}00, x_1\overline{x_0}01, x_1\overline{x_0}11, x_1\overline{x_0}10\}$, we list different source nodes of the *source* paths in Table 2. □

We have proposed the *source* paths with routing direction from one source node in $u \in CQ_4^{x_1x_0}$ to a set of ending nodes in $D \subseteq CQ_4^{\overline{x_1}x_0}$ or $D \subseteq CQ^{x_1\overline{x_0}}$. Notice that the set of ending nodes in the case 1 and case 2 of Lemma 3 are different direction from the source node. For example, from the source node $u = 0000$ to the ending nodes of $P_{1,S}$ in the case 1 and case 2 of Lemma 3 are 1011 and 0100, respectively. Thus, the *source* paths have four mutually independent paths in CQ_4 from any one source node u to four ending nodes $v \in D$ in adjacent CQ_2. Next, we will consider the *destination* paths as follows.

Remark 1. Let $u, v \in CQ_4$, D be a set of starting nodes in $CQ_4^{x_1x_0}$ and v be a destination node in $CQ_4^{y_1y_0}$. There exists four mutually independent paths in CQ_4 from nodes $u \in D$ to v if $CQ_4(x_1x_0)$ and $CQ_4(y_1y_0)$ are adjacent subgraphs.

Consequently, the *destination* paths are given by reverse-order of the *source* paths. Then we propose four mutually independent paths to be used in the *intermediate* paths. We exploit the same edges' dimension sequences to represent each of *intermediate* paths, and we list the *intermediate* paths from different starting nodes to ending nodes in Table 3. There are going to be used in Theorem 2.

Table 1. The edges' dimension from $u \in CQ_4^{x_1x_0}$ to $D \in CQ_4^{\bar{x}_1x_0}$

source node	edges' dimension	ending node
$x_1x_0 00$	021202130102101	$\bar{x}_1x_0 11$
	101201030102010	$\bar{x}_1x_0 01$
	201210231012101	$\bar{x}_1x_0 00$
	301201030102103	$\bar{x}_1x_0 10$
$x_1x_0 01$	021202130102101	$\bar{x}_1x_0 00$
	101201030102010	$\bar{x}_1x_0 10$
	201210231012101	$\bar{x}_1x_0 11$
	301201030102103	$\bar{x}_1x_0 01$
$x_1x_0 11$	021202130102101	$\bar{x}_1x_0 01$
	101201030102010	$\bar{x}_1x_0 11$
	201210231012101	$\bar{x}_1x_0 10$
	301201030102103	$\bar{x}_1x_0 00$
$x_1x_0 10$	021202130102101	$\bar{x}_1x_0 10$
	101201030102010	$\bar{x}_1x_0 00$
	201210231012101	$\bar{x}_1x_0 01$
	301201030102103	$\bar{x}_1x_0 11$

Remark 2. Let D_1 be a set of starting nodes in $CQ_4^{x_1x_0}$ and D_2 be a set of ending nodes in $CQ_4^{y_1y_0}$. There exists four mutually independent paths in CQ_4 from nodes in D_1 to nodes in D_2 if $CQ_4(x_1x_0)$ and $CQ_4(y_1y_0)$ are adjacent subgraphs.

The following theorem shows that there exists four mutually independent Hamiltonian paths in CQ_5 from u to v.

Theorem 1. *Let u and v be two distinct nodes of CQ_5. There are four mutually independent Hamiltonian paths in CQ_5 from u to v.*

Proof: Let u be the source node and v be the destination node. We will establish the mutually independent Hamiltonian paths of crossed cube by $P_i = P_{i,S} \cup \{e_L\} \cup P_{i,D}$. The *source* paths and the *destination* paths are shown in Lemma 3 and Remark 1. Assume $x_2x_1x_0$ and $y_2y_1y_0$ are two binary strings, we first let $u = u_4u_3u_2 00$ be source node in $CQ_5^{x_2x_1x_0}$ and $v = v_4v_3v_2 00$ be destination node in $CQ_5^{y_2y_1y_0}$, where

Table 2. The edges' dimension from $u \in CQ_4^{x_1x_0}$ to $D \in CQ^{x_1\bar{x}_0}$

source node	edges' dimension	ending node
$x_1x_0 00$	031303120103101	$x_1\bar{x}_0 11$
	101301020103010	$x_1\bar{x}_0 01$
	201301020103102	$x_1\bar{x}_0 10$
	301310321013101	$x_1\bar{x}_0 00$
$x_1x_0 01$	031303120103101	$x_1\bar{x}_0 00$
	101301020103010	$x_1\bar{x}_0 10$
	201301020103102	$x_1\bar{x}_0 01$
	301310321013101	$x_1\bar{x}_0 11$
$x_1x_0 11$	031303120103101	$x_1\bar{x}_0 01$
	101301020103010	$x_1\bar{x}_0 11$
	201301020103102	$x_1\bar{x}_0 00$
	301310321013101	$x_1\bar{x}_0 10$
$x_1x_0 10$	031303120103101	$x_1\bar{x}_0 10$
	101301020103010	$x_1\bar{x}_0 00$
	201301020103102	$x_1\bar{x}_0 11$
	301310321013101	$x_1\bar{x}_0 01$

$CQ_5(x_2x_1x_0)$ and $CQ_5(y_2y_1y_0)$ are adjacent subgraphs.

Case 1. $u \in CQ_5^{x_2}$ and $v \in CQ_5^{\bar{x}_2}$.

Subcase 1.1. $v \in CQ_5^{\bar{x}_2x_1x_0}$.

The *source* paths can look up the Table 1 from source node $u \in CQ_5^{x_2x_1x_0}$ to a set of ending nodes in $CQ_5^{x_2\bar{x}_1x_0}$ and the *destination* paths can look up the reverse-order of Table 1 from a set of starting nodes in $CQ_5^{\bar{x}_2\bar{x}_1x_0}$ to destination node $v \in CQ_5^{\bar{x}_2x_1x_0}$. There exists four mutually independent Hamiltonian paths as follows.

The first path P_1:

$P_{1,S} = (0,2,1,2,0,2,1,3,0,1,0,2,1,0,1)$, $\{e_L\} = \{4\}$,

$P_{1,D} = (0,1,0,2,0,1,0,3,0,1,0,2,1,0,1)$.

The second path P_2:

$P_{2,S} = (1,0,1,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_L\} = \{4\}$,

$P_{2,D} = (1,0,1,2,0,1,0,3,1,2,0,2,1,2,0)$.

The third path P_3:

$P_{3,S} = (2,0,1,2,1,0,2,3,1,0,1,2,1,0,1)$, $\{e_L\} = \{4\}$,

Table 3. The edges' dimension from $D_1 \in CQ_4^{x_1 x_0}$ to $D_2 \in CQ_4^{\overline{x_1} x_0}$

starting node	edges' dimension	ending node
$x_1x_0 00$	010201031012101	$\overline{x_1}x_0 00$
$x_1x_0 10$	010201031012101	$\overline{x_1}x_0 10$
$x_1x_0 11$	010201031012101	$\overline{x_1}x_0 01$
$x_1x_0 01$	010201031012101	$\overline{x_1}x_0 11$

$P_{3,D} = (1,0,1,2,1,0,1,3,2,0,1,2,1,0,2)$.

The fourth path P_4:

$P_{4,S} = (3,0,1,2,0,1,0,3,0,1,0,2,1,0,3)$, $\{e_L\} = \{4\}$,

$P_{4,D} = (3,0,1,2,0,1,0,3,0,1,0,2,1,0,3)$.

Subcase 1.2. $v \in CQ_5^{\overline{x_2}\,\overline{x_1}\,\overline{x_0}}$.

The *source* paths can look up the Table 1 from source node $u \in CQ_5^{x_2 x_1 x_0}$ to four ending nodes $CQ_5^{x_2 \overline{x_1} x_0}$ and the *destination* paths can look up the reverse-order of Table 2 from four starting nodes $CQ_5^{\overline{x_2}\,\overline{x_1} x_0}$ to destination node $v \in CQ_5^{\overline{x_2}\,\overline{x_1}\,\overline{x_0}}$. There exists four mutually independent Hamiltonian paths as follows.

$P_{1,S} = (0,2,1,2,0,2,1,3,0,1,0,2,1,0,1)$, $\{e_L\} = \{4\}$,

$P_{1,D} = (0,1,0,3,0,1,0,2,0,1,0,3,1,0,1)$.

$P_{2,S} = (1,0,1,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_L\} = \{4\}$,

$P_{2,D} = (1,0,1,3,0,1,0,2,1,3,0,3,1,3,0)$.

$P_{3,S} = (2,0,1,2,1,0,2,3,1,0,1,2,1,0,1)$, $\{e_L\} = \{4\}$,

$P_{3,D} = (1,0,1,3,1,0,1,2,3,0,1,3,1,0,3)$.

$P_{4,S} = (3,0,1,2,0,1,0,3,0,1,0,2,1,0,3)$, $\{e_L\} = \{4\}$,

$P_{4,D} = (2,0,1,3,0,1,0,2,0,1,0,3,1,0,2)$.

Subcase 1.3. $v \in CQ_5^{\overline{x_2} x_1 \overline{x_0}}$.

The *source* paths can look up the Table 2 from source node $u \in CQ_5^{x_2 x_1 x_0}$ to four ending nodes $CQ_5^{x_2 x_1 \overline{x_0}}$ and the *destination* paths can look up the reverse-order of Table 1 from four starting nodes $CQ_5^{\overline{x_2}\,\overline{x_1}\,\overline{x_0}}$ to destination node $v \in CQ_5^{\overline{x_2} x_1 \overline{x_0}}$. There exists four mutually independent Hamiltonian paths as follows.

$P_{1,S} = (0,3,1,3,0,3,1,2,0,1,0,3,1,0,1)$, $\{e_L\} = \{4\}$,

$P_{1,D} = (0,1,0,2,0,1,0,3,0,1,0,2,1,0,1)$.

$P_{2,S} = (1,0,1,3,0,1,0,2,0,1,0,3,0,1,0)$, $\{e_L\} = \{4\}$,

$P_{2,D} = (1,0,1,2,0,1,0,3,1,2,0,2,1,2,0)$.

$P_{3,S} = (2,0,1,3,0,1,0,2,0,1,0,3,1,0,2)$, $\{e_L\} = \{4\}$,

$P_{3,D} = (3,0,1,2,0,1,0,3,0,1,0,2,1,0,3)$.

$P_{4,S} = (3,0,1,3,1,0,3,2,1,0,1,3,1,0,1)$, $\{e_L\} = \{4\}$,

$P_{4,D} = (1,0,1,2,1,0,1,3,2,0,1,2,1,0,2)$.

Subcase 1.4. $v \in CQ_5^{\overline{x_2}\,\overline{x_1} x_0}$.

The *source* paths can look up the Table 2 from source node $u \in CQ_5^{x_2 x_1 x_0}$ to four ending nodes $CQ_5^{x_2 x_1 \overline{x_0}}$ and the *destination* paths can look up the reverse-order of Table 2 from four starting nodes $CQ_5^{\overline{x_2}\,\overline{x_1}\,\overline{x_0}}$ to destination node $v \in CQ_5^{\overline{x_2}\,\overline{x_1} x_0}$. There exists four mutually independent Hamiltonian paths as follows.

$P_{1,S} = (0,3,1,3,0,3,1,2,0,1,0,3,1,0,1)$, $\{e_L\} = \{4\}$,

$P_{1,D} = (0,1,0,3,0,1,0,2,0,1,0,3,1,0,1)$.

$P_{2,S} = (1,0,1,3,0,1,0,2,0,1,0,3,0,1,0)$, $\{e_L\} = \{4\}$,

$P_{2,D} = (1,0,1,3,0,1,0,2,1,3,0,3,1,3,0)$.

$P_{3,S} = (2,0,1,3,0,1,0,2,0,1,0,3,1,0,2)$, $\{e_L\} = \{4\}$,

$P_{3,D} = (2,0,1,3,0,1,0,2,0,1,0,3,1,0,2)$.

$P_{4,S} = (3,0,1,3,1,0,3,2,1,0,1,3,1,0,1)$, $\{e_L\} = \{4\}$,

$P_{4,D} = (1,0,1,3,1,0,1,2,3,0,1,3,1,0,3)$.

Case 2. $u \in CQ_5^{x_2}$ and $v \in CQ_5^{x_2}$.

Subcase 2.1. $v \in CQ_5^{x_2 x_1 \overline{x_0}}$.

In this case, we interchange the edge dimension "2" with "4" of subcase 1.1.

$P_{1,S} = (0,4,1,4,0,4,1,3,0,1,0,4,1,0,1)$, $\{e_L\} = \{2\}$,

$P_{1,D} = (0,1,0,4,0,1,0,3,0,1,0,4,1,0,1)$.

$P_{2,S} = (1,0,1,4,0,1,0,3,0,1,0,4,0,1,0)$, $\{e_L\} = \{2\}$,

$P_{2,D} = (1,0,1,4,0,1,0,3,1,4,0,4,1,4,0)$.

$P_{3,S} = (4,0,1,4,1,0,4,3,1,0,1,4,1,0,1)$, $\{e_L\} = \{2\}$,

$P_{3,D} = (1,0,1,4,1,0,1,3,4,0,1,4,1,0,4)$.

$P_{4,S} = (3,0,1,4,0,1,0,3,0,1,0,4,1,0,3)$, $\{e_L\} = \{2\}$,

$P_{4,D} = (3,0,1,4,0,1,0,3,0,1,0,4,1,0,3)$.

Subcase 2.2. $v \in CQ_5^{x_2 \overline{x_1}\,\overline{x_0}}$.

In this case, we interchange the edge dimension "2" with "4" of subcase 1.4.

$P_{1,S} = (0,3,1,3,0,3,1,4,0,1,0,3,1,0,1)$, $\{e_L\} = \{2\}$,

$P_{1,D} = (0,1,0,3,0,1,0,4,0,1,0,3,1,0,1)$.

$P_{2,S} = (1,0,1,3,0,1,0,4,0,1,0,3,0,1,0)$, $\{e_L\} = \{2\}$,

$P_{2,D} = (1,0,1,3,0,1,0,4,1,3,0,3,1,3,0)$.

$P_{3,S} = (4,0,1,3,0,1,0,4,0,1,0,3,1,0,4)$, $\{e_L\} = \{2\}$,
$P_{3,D} = (4,0,1,3,0,1,0,4,0,1,0,3,1,0,4)$.
$P_{4,S} = (3,0,1,3,1,0,3,4,1,0,1,3,1,0,1)$, $\{e_L\} = \{2\}$,
$P_{4,D} = (1,0,1,3,1,0,1,4,3,0,1,3,1,0,3)$.

Subcase 2.3. $v \in CQ_5^{x_2\bar{x}_1x_0}$.

In this case, we interchange the edge dimension "3" with "4" of subcase 1.1.

$P_{1,S} = (0,2,1,2,0,2,1,4,0,1,0,2,1,0,1)$, $\{e_L\} = \{3\}$,
$P_{1,D} = (0,1,0,2,0,1,0,4,0,1,0,2,1,0,1)$.
$P_{2,S} = (1,0,1,2,0,1,0,4,0,1,0,2,0,1,0)$, $\{e_L\} = \{3\}$,
$P_{2,D} = (1,0,1,2,0,1,0,4,1,2,0,2,1,2,0)$.
$P_{3,S} = (2,0,1,2,1,0,2,4,1,0,1,2,1,0,1)$, $\{e_L\} = \{3\}$,
$P_{3,D} = (1,0,1,2,1,0,1,4,2,0,1,2,1,0,2)$.
$P_{4,S} = (4,0,1,2,0,1,0,4,0,1,0,2,1,0,4)$, $\{e_L\} = \{3\}$,
$P_{4,D} = (4,0,1,2,0,1,0,4,0,1,0,2,1,0,4)$.

Subcase 2.4. $v \in CQ_5^{x_2x_1x_0}$.

In this subcase, the four mutually independent Hamiltonian paths are obtained by an exhaustive search. Three cases for v are distinguished.

Subcase 2.4.1. $v = x_2x_1x_0 01$.

$P_{1,S} = (1,0,2,0,1,0,3,1,2,0,2,1,2,4,1)$, $\{e_L\} = \{2\}$,
$P_{1,D} = (1,0,1,2,3,0,2,0,1,0,2,0,3,4,3)$.
$P_{2,S} = (2,0,1,0,3,0,2,1,2,0,2,1,3,4,1)$, $\{e_L\} = \{2\}$,
$P_{2,D} = (1,0,1,2,3,0,2,0,1,0,2,0,3,4,1)$.
$P_{3,S} = (3,0,2,0,1,0,2,0,3,0,2,0,1,4,1)$, $\{e_L\} = \{2\}$,
$P_{3,D} = (1,0,1,2,3,0,2,0,1,0,2,0,3,4,2)$.
$P_{4,S} = (4,1,2,0,2,3,1,0,1,2,0,1,0,4,0)$, $\{e_L\} = \{1\}$,
$P_{4,D} = (0,2,0,3,0,1,0,2,1,0,1,4,0,2,4)$.

Subcase 2.4.2 $v = x_2x_1x_0 11$.

$P_{1,S} = (0,2,1,0,1,3,0,2,1,2,0,2,1,4,0)$, $\{e_L\} = \{1\}$,
$P_{1,D} = (0,2,0,1,0,3,1,2,0,2,1,2,0,4,0)$.
$P_{2,S} = (1,2,1,0,1,3,0,2,1,2,0,2,1,4,0)$, $\{e_L\} = \{1\}$,
$P_{2,D} = (0,2,1,0,1,3,1,2,1,0,2,1,2,4,1)$.
$P_{3,S} = (2,3,0,1,0,3,2,3,1,0,1,3,2,4,0)$, $\{e_L\} = \{1\}$,
$P_{3,D} = (3,0,1,0,2,1,0,1,3,0,1,0,2,4,2)$.
$P_{4,S} = (3,1,3,2,1,3,1,0,3,1,3,2,3,4,2)$, $\{e_L\} = \{1\}$,
$P_{4,D} = (2,0,1,2,1,3,1,2,1,0,2,1,2,4,3)$.

Subcase 2.4.3 $v = x_2x_1x_0 10$.

$P_{1,S} = (0,2,0,1,0,2,3,1,2,1,0,1,2,4,0)$, $\{e_L\} = \{2\}$,
$P_{1,D} = (0,1,0,2,3,0,2,1,0,1,2,0,3,4,3)$.
$P_{2,S} = (2,1,3,1,0,1,3,1,2,3,0,1,0,4,0)$, $\{e_L\} = \{2\}$,
$P_{2,D} = (0,1,0,2,0,3,2,0,2,1,2,0,2,4,0)$.
$P_{3,S} = (3,1,0,1,3,1,2,3,0,1,0,3,0,4,0)$, $\{e_L\} = \{2\}$,
$P_{3,D} = (0,1,0,2,3,0,2,1,0,1,2,0,3,4,2)$.
$P_{4,S} = (4,0,1,2,1,3,1,0,2,0,1,0,2,4,1)$, $\{e_L\} = \{0\}$,
$P_{4,D} = (1,2,1,3,2,1,2,0,1,2,1,4,1,2,4)$.

Clearly, each of path in P_i are independent, so we have constructed four mutually independent Hamiltonian paths from the source node $u = u_4u_3u_2 00$ to the destination node $v = v_4v_3v_2 00$. Note that we can use the proposed edges' dimension to construct the others type of the *destination* paths $P_{i,D}$ from the source node $u = u_4u_3u_2 00$ to $v = v_4v_3v_2 01$ 、 $v = v_4v_3v_2 11$ or $v = v_4v_3v_2 10$, respectively. Obviously, there are four mutually independent Hamiltonian paths from $u = u_4u_3u_2 00$ to $v = v_4v_3v_2v_1v_0$.

Next, we construct the source node $u = u_4u_3u_2 01$ to destination nodes $v = v_4v_3v_2 00$, $v = v_4v_3v_2 01$ 、 $v = v_4v_3v_2 11$ or $v = v_4v_3v_2 10$, respectively. Using the same edges' dimension sequences as source node $u = u_4u_3u_2 00$ to $v = v_4v_3v_2v_1v_0$, we also obtain the mutually independent Hamiltonian paths from u to v. As mentioned before, we can use the proposed edges' dimension to construct the others type of the *source* paths $P_{i,S}$ from each source node $u = u_4u_3u_2 11$ or $u = u_4u_3u_2 10$ to destination node $v = v_4v_3v_2v_1v_0$. Thus, we have established the mutually independent Hamiltonian paths of crossed cubes by $P_i = P_{i,S} \cup \{e_L\} \cup P_{i,D}$. Clearly, each of path in P_i are independently, so there are four mutually independent Hamiltonian paths in CQ_5 from any two distinct nodes u and v. □

Huang et. al. [14] proved that CQ_n is $(n-2)$-Hamiltonian and $(n-3)$-Hamiltonian connected, the faulty CQ_n is still Hamiltonian with $n-2$ faults.

Lemma 4[14]. *The crossed cube* CQ_n *is* $(n-2)$*-Hamiltonian and* $(n-3)$*-Hamiltonian connected for* $n \geq 3$.

An *n*-dimensional crossed cube is

Hamiltonian connected [16], i.e., a Hamiltonian path can be constructed from any node to any other node in the network. A graph G is Hamiltonian connected if there exists a Hamiltonian path joining any two distinct nodes. Therefore, the Hamiltonian connected property is used in Theorem 2. The following theorem is proven that there exists four mutually independent Hamiltonian paths from u and v in CQ_n with $n>5$.

Theorem 2. *Let u and v be two distinct nodes of CQ_n. There are four mutually independent Hamiltonian paths from u to v in CQ_n for $n>5$.*

Proof: By Theorem 1, we have established four mutually independent Hamiltonian paths in CQ_5 from any two distinct nodes. In this theorem, we establish the mutually independent Hamiltonian paths of crossed cubes by the form of $P_i = P_{i,S} \cup \{e_{L1}\} \cup P_{i,I} \cup \{e_{L2}\} \cup P_{i,D}$. By Lemma 4, there exists a Hamiltonian path $P' = (u^1, u^2, ..., u^{2^{n-4}})$ from arbitrary source node to any destination node in CQ_{n-4}. Notably, each node of P' is a subcube CQ_4 of CQ_n. The source node can be extended to the *source* paths and the destination node can be extended to the *destination* paths as shown in Tables 1 and 2. The *source* paths and the *destination* paths are length 16. There are $2^{n-4}-2$ intermediate nodes u^j for $j \neq 1, 2^{n-4}$ in P' can be extended to the *intermediate* paths with length $(2^{n-4}-2)\times 16$. Since the nodes u^1 and u^2 are adjacent nodes, and nodes $u^{2^{n-4}-1}$ and $u^{2^{n-4}}$ are also adjacent in CQ_{n-4}. Therefore, the *source* paths and the *destination* paths can be exactly generated by Tables 1 and 2. Furthermore, nodes u^i and u^{i+1} are adjacent nodes in P' for $2 \le i \le 2^{n-4}-2$. Since there are four ending nodes located at the same CQ_4 in *source* paths and four starting nodes located at the same CQ_4 in *destination* paths, the *intermediate* paths must be four starting nodes and four ending nodes. Also, by Table 3, we have four mutually independent paths in the same CQ_4. Thus, we can construct the *intermediate* paths along the intermediate nodes u^j for $j \neq 1, 2^{n-4}$ in P'.

Assume $p_{n-5}(x)x_2x_1x_0 00$ and $p_{n-5}(y)y_2y_1y_0 00$ are two binary strings, we first let $u = p_{n-5}(u)u_2u_1u_0 00$ be source node in $CQ_n^{p_{n-5}(x)x_2x_1x_0}$ and $v = p_{n-5}(v)v_2v_1v_0 00$ be destination node in $CQ_n^{p_{n-5}(y)y_2y_1y_0}$, where $CQ_n(p_{n-5}(x)x_2x_1x_0)$ and $CQ_n(p_{n-5}(y)y_2y_1y_0)$ are adjacent subgraphs. This proof is distinguished two cases as follows.

Case 1. $u \in CQ_n^{p_{n-5}(x)x_2}$ and $v \in CQ_n^{p_{n-5}(x)\overline{x_2}}$.

In this case, the *source* paths and the *destination* paths are same as the case 1 of Theorem 1. The destination node v is divided into in the following cases.

Subcase 1.1. $v \in CQ_n^{p_{n-5}(x)\overline{x_2}x_1x_0}$.

$P_{1,S} = (0,2,1,2,0,2,1,3,0,1,0,2,1,0,1)$, $\{e_{L1}\} = \{4\}$,

$P_{1,I} = (0,1,0,2,0,1,0,3,1,0,1,2,1,0,1)$, $\{e_{L2}\} = \{4\}$,

$P_{1,D} = (0,1,0,2,0,1,0,3,0,1,0,2,1,0,1)$.

$P_{2,S} = (1,0,1,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L1}\} = \{4\}$,

$P_{2,I} = (0,1,0,2,0,1,0,3,1,0,1,2,1,0,1)$, $\{e_{L2}\} = \{4\}$,

$P_{2,D} = (1,0,1,2,0,1,0,3,1,2,0,2,1,2,0)$.

$P_{3,S} = (2,0,1,2,1,0,2,3,1,0,1,2,1,0,1)$, $\{e_{L1}\} = \{4\}$,

$P_{3,I} = (0,1,0,2,0,1,0,3,1,0,1,2,1,0,1)$, $\{e_{L2}\} = \{4\}$,

$P_{3,D} = (1,0,1,2,1,0,1,3,2,0,1,2,1,0,2)$.

$P_{4,S} = (3,0,1,2,0,1,0,3,0,1,0,2,1,0,3)$, $\{e_{L1}\} = \{4\}$,

$P_{4,I} = (0,1,0,2,0,1,0,3,1,0,1,2,1,0,1)$, $\{e_{L2}\} = \{4\}$,

$P_{4,D} = (3,0,1,2,0,1,0,3,0,1,0,2,1,0,3)$.

Subcase 1.2. $v \in CQ_n^{p_{n-5}(x)\overline{x_2}\,\overline{x_1}\,\overline{x_0}}$.

$P_{1,S} = (0,2,1,2,0,2,1,3,0,1,0,2,1,0,1)$, $\{e_{L1}\} = \{4\}$,

$P_{1,I} = (0,1,0,2,0,1,0,3,1,0,1,2,1,0,1)$, $\{e_{L2}\} = \{4\}$,

$P_{1,D} = (0,1,0,3,0,1,0,2,0,1,0,3,1,0,1)$.

$P_{2,S} = (1,0,1,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L1}\} = \{4\}$,

$P_{2,I} = (0,1,0,2,0,1,0,3,1,0,1,2,1,0,1)$, $\{e_{L2}\} = \{4\}$,

$P_{2,D} = (1,0,1,3,0,1,0,2,1,3,0,3,1,3,0)$.

$P_{3,S} = (2,0,1,2,1,0,2,3,1,0,1,2,1,0,1)$, $\{e_{L1}\} = \{4\}$,

$P_{3,I} = (0,1,0,2,0,1,0,3,1,0,1,2,1,0,1)$, $\{e_{L2}\} = \{4\}$,

$P_{3,D} = (1,0,1,3,1,0,1,2,3,0,1,3,1,0,3)$.

$P_{4,S} = (3,0,1,2,0,1,0,3,0,1,0,2,1,0,3)$, $\{e_{L1}\} = \{4\}$,

$P_{4,I} = (0,1,0,2,0,1,0,3,1,0,1,2,1,0,1)$, $\{e_{L2}\} = \{4\}$,

$P_{4,D} = (2,0,1,3,0,1,0,2,0,1,0,3,1,0,2)$.

Subcase 1.3. $v \in CQ_n^{p_{n-5}(x)\overline{x_2}x_1\overline{x_0}}$.

$P_{1,S}=(0,3,1,3,0,3,1,2,0,1,0,3,1,0,1)$, $\{e_{L1}\}=\{4\}$,
$P_{1,I}=(0,1,0,2,0,1,0,3,1,0,1,2,1,0,1)$, $\{e_{L2}\}=\{4\}$,
$P_{1,D}=(0,1,0,2,0,1,0,3,0,1,0,2,1,0,1)$.
$P_{2,S}=(1,0,1,3,0,1,0,2,0,1,0,3,0,1,0)$, $\{e_{L1}\}=\{4\}$,
$P_{2,I}=(0,1,0,2,0,1,0,3,1,0,1,2,1,0,1)$, $\{e_{L2}\}=\{4\}$,
$P_{2,D}=(1,0,1,2,0,1,0,3,1,2,0,2,1,2,0)$.
$P_{3,S}=(2,0,1,3,0,1,0,2,0,1,0,3,1,0,2)$, $\{e_{L1}\}=\{4\}$,
$P_{3,I}=(0,1,0,2,0,1,0,3,1,0,1,2,1,0,1)$, $\{e_{L2}\}=\{4\}$,
$P_{3,D}=(3,0,1,2,0,1,0,3,0,1,0,2,1,0,3)$.
$P_{4,S}=(3,0,1,3,1,0,3,2,1,0,1,3,1,0,1)$, $\{e_{L1}\}=\{4\}$,
$P_{4,I}=(0,1,0,2,0,1,0,3,1,0,1,2,1,0,1)$, $\{e_{L2}\}=\{4\}$,
$P_{4,D}=(1,0,1,2,1,0,1,3,2,0,1,2,1,0,2)$.

Subcase 1.4. $v\in CQ_n^{p_{n-5}(x)\overline{x_2}\,\overline{x_1}x_0}$.
$P_{1,S}=(0,3,1,3,0,3,1,2,0,1,0,3,1,0,1)$, $\{e_{L1}\}=\{4\}$,
$P_{1,I}=(0,1,0,2,0,1,0,3,1,0,1,2,1,0,1)$, $\{e_{L2}\}=\{4\}$,
$P_{1,D}=(0,1,0,3,0,1,0,2,0,1,0,3,1,0,1)$.
$P_{2,S}=(1,0,1,3,0,1,0,2,0,1,0,3,0,1,0)$, $\{e_{L1}\}=\{4\}$,
$P_{2,I}=(0,1,0,2,0,1,0,3,1,0,1,2,1,0,1)$, $\{e_{L2}\}=\{4\}$,
$P_{2,D}=(1,0,1,3,0,1,0,2,1,3,0,3,1,3,0)$.
$P_{3,S}=(2,0,1,3,0,1,0,2,0,1,0,3,1,0,2)$, $\{e_{L1}\}=\{4\}$,
$P_{3,I}=(0,1,0,2,0,1,0,3,1,0,1,2,1,0,1)$, $\{e_{L2}\}=\{4\}$,
$P_{3,D}=(2,0,1,3,0,1,0,2,0,1,0,3,1,0,2)$.
$P_{4,S}=(3,0,1,3,1,0,3,2,1,0,1,3,1,0,1)$, $\{e_{L1}\}=\{4\}$,
$P_{4,I}=(0,1,0,2,0,1,0,3,1,0,1,2,1,0,1)$, $\{e_{L2}\}=\{4\}$,
$P_{4,D}=(1,0,1,3,1,0,1,2,3,0,1,3,1,0,3)$.

Case 2. $u\in CQ_n^{p_{n-5}(x)x_2}$ and $v\in CQ_n^{p_{n-5}(x)x_2}$.
In this case, the routing sequence of *source* paths and the *destination* paths are same as the case 2 of Theorem 1. We choose the edge dimension "4" to connect *intermediate* paths in $P_{i,S}$ and $P_{i,D}$, respectively. For all *intermediate* paths, there is the same sequence of edge's dimension to join each other. The destination node v is divided into in the following cases.

Subcase 2.1. $v\in CQ_n^{p_{n-5}(x)x_2x_1\overline{x_0}}$.
$P_{1,S}=(0,4,1,4,0,4,1,3,0,1,0)$, $\{e_{L1}\}=\{4\}$,
$P_{1,I}=(0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\}=\{4\}$,
$P_{1,D}=(1,0,1,2,0,1,0,4,0,1,0,3,0,1,0,4,1,0,1)$.
$P_{2,S}=(1,0,1,4,0,1,0,3,0,1,0)$, $\{e_{L1}\}=\{4\}$,
$P_{2,I}=(0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\}=\{4\}$,
$P_{2,D}=(0,1,0,2,1,0,1,4,0,1,0,3,1,4,0,4,1,4,0)$.
$P_{3,S}=(4,0,1,4,1,0,4,3,1,0,1)$, $\{e_{L1}\}=\{4\}$,
$P_{3,I}=(0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\}=\{4\}$,
$P_{3,D}=(1,0,1,2,1,0,1,4,1,0,1,3,4,0,1,4,1,0,4)$.
$P_{4,S}=(3,0,1,4,0,1,0,3,0,1,0)$, $\{e_{L1}\}=\{4\}$,
$P_{4,I}=(0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\}=\{4\}$,
$P_{4,D}=(1,0,3,2,3,0,1,4,0,1,0,3,0,1,0,4,1,0,3)$.

Subcase 2.2. $v\in CQ_n^{p_{n-5}(x)x_2\overline{x_1}\,\overline{x_0}}$.
$P_{1,S}=(0,3,1,3,0,3,1,4,0,1,0,3,1,0,1,2,0,1,0,3,0,1,0)$, $\{e_{L1}\}=\{4\}$, $P_{1,I}=(0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\}=\{4\}$, $P_{1,D}=(0,1,0,3,1,0,1)$.
$P_{2,S}=(1,0,1,3,0,1,0,4,0,1,0,3,0,1,0,2,1,0,1,3,0,1,0)$, $\{e_{L1}\}=\{4\}$, $P_{2,I}=(0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\}=\{4\}$, $P_{2,D}=(1,3,0,3,1,3,0)$.
$P_{3,S}=(4,0,1,3,0,1,0,4,0,1,0,3,1,0,4,2,4,0,1,3,0,1,0)$, $\{e_{L1}\}=\{4\}$, $P_{3,I}=(0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\}=\{4\}$, $P_{3,D}=(0,1,0,3,1,0,4)$.
$P_{4,S}=(3,0,1,3,1,0,3,4,1,0,1,3,1,0,1,2,1,0,1,3,1,0,1)$, $\{e_{L1}\}=\{4\}$, $P_{4,I}=(0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\}=\{4\}$, $P_{4,D}=(3,0,1,3,1,0,3)$.

Subcase 2.3. $v\in CQ_n^{p_{n-5}(x)x_2\overline{x_1}x_0}$.
$P_{1,S}=(0,2,1,2,0,2,1,4,0,1,0,2,1,0,1,3,0,1,0,2,0,1,0)$, $\{e_{L1}\}=\{4\}$, $P_{1,I}=(0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\}=\{4\}$, $P_{1,D}=(0,1,0,2,1,0,1)$.
$P_{2,S}=(1,0,1,2,0,1,0,4,0,1,0,2,0,1,0,3,1,0,1,2,0,1,0)$, $\{e_{L1}\}=\{4\}$, $P_{2,I}=(0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\}=\{4\}$, $P_{2,D}=(1,2,0,2,1,2,0)$.
$P_{3,S}=(2,0,1,2,1,0,2,4,1,0,1,2,1,0,1,3,1,0,1,2,1,0,1)$, $\{e_{L1}\}=\{4\}$, $P_{3,I}=(0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\}=\{4\}$, $P_{3,D}=(2,0,1,2,1,0,2)$.
$P_{4,S}=(4,0,1,2,0,1,0,4,0,1,0,2,1,0,4,3,4,0,1,2,0,1,0)$, $\{e_{L1}\}=\{4\}$, $P_{4,I}=(0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\}=\{4\}$, $P_{4,D}=(0,1,0,2,1,0,4)$.

Subcase 2.4. $v\in CQ_n^{p_{n-5}(x)x_2x_1x_0}$.
Subcase 2.4.1. $v=p_{n-5}(x)x_2x_1x_0 01$.

$P_{1,S} = (1,0,2,0,1,0,3,1,2,0,2,1,2)$, $\{e_{L1}\} = \{4\}$,

$P_{1,I} = (0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\} = \{4\}$,

$P_{1,D} = (1,2,1,0,1,2,3,0,2,0,1,0,2,0,3,4,3)$.

$P_{2,S} = (2,0,1,0,3,0,2,1,2,0,2,1,3)$, $\{e_{L1}\} = \{4\}$,

$P_{2,I} = (0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\} = \{4\}$,

$P_{2,D} = (1,2,1,0,1,2,3,0,2,0,1,0,2,0,3,4,1)$.

$P_{3,S} = (3,0,2,0,1,0,2,0,3,0,2,0,1)$, $\{e_{L1}\} = \{4\}$,

$P_{3,I} = (0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\} = \{4\}$,

$P_{3,D} = (1,2,1,0,1,2,3,0,2,0,1,0,2,0,3,4,2)$.

$P_{4,S} = (4,1,2,0,2,3,1,0,1,2,0,1,0)$, $\{e_{L1}\} = \{4\}$,

$P_{4,I} = (0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\} = \{4\}$,

$P_{4,D} = (0,1,0,2,0,3,0,1,0,2,1,0,1,4,0,2,4)$.

Subcase 2.4.2 $v = p_{n-5}(x)x_2x_1x_0 11$.

$P_{1,S} = (0,2,1,0,1,3,0,2,1,2,0,2,1)$, $\{e_{L1}\} = \{4\}$,

$P_{1,I} = (0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\} = \{4\}$,

$P_{1,D} = (0,1,0,2,0,1,0,3,1,2,0,2,1,2,0,4,0)$.

$P_{2,S} = (1,2,1,0,1,3,0,2,1,2,0,2,1)$, $\{e_{L1}\} = \{4\}$,

$P_{2,I} = (0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\} = \{4\}$,

$P_{2,D} = (0,1,0,2,1,0,1,3,1,2,1,0,2,1,2,4,1)$.

$P_{3,S} = (2,3,0,1,0,3,2,3,1,0,1,3,2)$, $\{e_{L1}\} = \{4\}$,

$P_{3,I} = (0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\} = \{4\}$,

$P_{3,D} = (0,1,3,0,1,0,2,1,0,1,3,0,1,0,2,4,2)$.

$P_{4,S} = (3,1,3,2,1,3,1,0,3,1,3,2,3)$, $\{e_{L1}\} = \{4\}$,

$P_{4,I} = (0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\} = \{4\}$,

$P_{4,D} = (2,1,2,0,1,2,1,3,1,2,1,0,2,1,2,4,3)$.

Subcase 2.4.3 $v = p_{n-5}(x)x_2x_1x_0 10$.

$P_{1,S} = (0,2,0,1,0,2,3,1,2,1,0,1,2)$, $\{e_{L1}\} = \{4\}$,

$P_{1,I} = (0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\} = \{4\}$,

$P_{1,D} = (0,2,0,1,0,2,3,0,2,1,0,1,2,0,3,4,3)$.

$P_{2,S} = (2,1,3,1,0,1,3,1,2,3,0,1,0)$, $\{e_{L1}\} = \{4\}$,

$P_{2,I} = (0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\} = \{4\}$,

$P_{2,D} = (0,2,0,1,0,2,0,3,2,0,2,1,2,0,2,4,0)$.

$P_{3,S} = (3,1,0,1,3,1,2,3,0,1,0,3,0)$, $\{e_{L1}\} = \{4\}$,

$P_{3,I} = (0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\} = \{4\}$,

$P_{3,D} = (0,2,0,1,0,2,3,0,2,1,0,1,2,0,3,4,2)$.

$P_{4,S} = (4,0,1,2,1,3,1,0,2,0,1,0,2)$, $\{e_{L1}\} = \{4\}$,

$P_{4,I} = (0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, $\{e_{L2}\} = \{4\}$,

$P_{4,D} = (1,0,1,2,1,3,2,1,2,0,1,2,1,4,1,2,4)$.

We have established four mutually independent Hamiltonian paths from the source node $u = p_{n-5}(u)u_4u_3u_2 00$ to the destination node $v = p_{n-5}(v)v_4v_3v_2 00$. Obviously, we can use the proposed edges' dimension to construct the others type of the *destination* paths $P_{i,D}$ from the source node $u = p_{n-5}(u)u_4u_3u_2 00$ to $v = p_{n-5}(v)v_4v_3v_2 01$ 、 $v = p_{n-5}(v)v_4v_3v_2 11$ or $v = p_{n-5}(v)v_4v_3v_2 10$, respectively.

Next, we also use the proposed edges' dimension to construct the source node $u = p_{n-5}(u)u_4u_3u_2 01$ to destination nodes $v = p_{n-5}(v)v_4v_3v_2 00$ 、 $v = p_{n-5}(v)v_4v_3v_2 01$ 、 $v = p_{n-5}(v)v_4v_3v_2 11$ or $v = p_{n-5}(v)v_4v_3v_2 10$, respectively. Using the same edges' dimension sequences as source node $u = p_{n-5}(u)u_4u_3u_2 00$ to $v = p_{n-5}(v)v_4v_3v_2v_1v_0$, we can construct the mutually independent Hamiltonian paths from u to v. Similarly, we can use the proposed edges' dimension to construct the others type of the *source* paths $P_{i,S}$ from one of starting node $u = p_{n-5}(u)u_2u_1u_0 11$ or $u = p_{n-5}(u)u_2u_1u_0 10$ to destination node $v = p_{n-5}(v)v_4v_3v_2v_1v_0$.

Thus, we have established the mutually independent Hamiltonian paths of crossed cubes by $P_i = P_{i,S} \cup \{e_{L1}\} \cup P_{i,I} \cup \{e_{L2}\} \cup P_{i,D}$. Clearly, each of path in P_i is independent, so there are four mutually independent Hamiltonian paths in CQ_n from any two distinct nodes u and v. □

Theorem 3. *Let u and v be two distinct nodes of CQ_n. There are four mutually independent Hamiltonian paths from u to v in CQ_n for $n \geq 5$.*

<00000,00001,00111,00101,00011,00010,00110,00100,01100,01101,01111,01110,01010,01000,01001,01011,

Table 4. Four mutually independent Hamiltonian paths from 00000 to 10000

11001,11000,11010,11011,11101,11100,11110,11111,10101,10100,10110,10111,10001,10011,10010,10000>
<00000,00010,00011,00001,00111,00110,00100,00101,01111,01110,01100,01101,01011,01010,10000,01001, 11011,11001,11000,11010,11110,11111,11101,11100,10100,10110,10010,10011,10101,10111,10001,10000>
<00000,00100,00101,00111,00001,00011,00010,00110,01110,01100,01101,01111,01001,01011,01010,01000, 11000,11010,11011,11001,11111,11101,11100,11110,10110,10010,10011,10001,10111,10101,10100,10000>
<00000,01000,01001,01011,01101,01100,01110,01111,00101,00100,00110,00111,00001,00011,00010,01010, 11010,10010,10011,10001,10111,10110,10100,10101,11111,11110,11100,11101,11011,11001,11000,10000>

Proof: By Theorems 1 and 2, we have established four mutually independent Hamiltonian paths from u to v in CQ_n with $n=5$ and $n>5$, respectively. Consequently, there are four mutually independent Hamiltonian paths from any two distinct nodes in CQ_n with $n \geq 5$. □

The mutually independent Hamiltonian paths can be used in many applications. For example, in communication networking, there are 4 pieces of data needed to be sent from u to v, and the data needed to be processed at every node (and the process takes time), then we want mutually independent Hamiltonian paths from u to v so that there will be no waiting at a processor. Table 4 is an example of four mutually independent Hamiltonian paths from $u=00000$ to $v=10000$ in CQ_5. For broadcasting a message, it can be devided into sub-messages transmitted via mutually independent Hamiltonian paths as shown in Table 4. The existence of mutually independent Hamiltonian paths is useful for communication algorithm.

Ⅳ. CONCLUSIONS

In this paper, the mutually independent paths of the crossed cubes have been addressed. We show that an n-dimensional crossed cube have four mutually independent Hamiltonian paths from any two distinct nodes u to v for $n \geq 5$. Moreover, we establish the routing method to generate four mutually independent Hamiltonian paths in CQ_n for $n \geq 5$. It indicates the mutually independent Hamiltonian paths can be applied to broadcast over wireless communication for the frequency channels assignment of backbone stations to the transmitters in such a way. In the future works, the focus of this issue is how to establish $n-1$ mutually independent Hamiltonian paths in CQ_n. These related studies will be interesting topics to be explored.

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