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短文 Short Note

Nonlinear H_∞ Control of the Inverted Pendulum

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ABSTRACT

In this paper, we present a computational procedure for an approximate solution of the Hamilton-Jacobi equation (HJE) which is employed to construct a nonlinear H_∞ controller for the inverted pendulum system. The closed-loop pendulum responses for both the nonlinear and the linear H_∞ controllers will be simulated and the robust stability/performance of the two controllers will also be compared.

Key words: nonlinear H_∞ control, energy dissipation, Hamilton-Jacobi equation, inverted pendulum

倒單擺之非線性 H_∞ 控制

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摘要

本論文提出了求漢米爾頓-傑克比(Hamilton-Jacobi)方程式近似解之計算法則，以建構非線性 H_∞ 控制器。文中介紹倒單擺系統之非線性及線性 H_∞ 控制器求解之方法，並附上閉迴路響應之系統模擬，以比較二者在強健穩定性及強健性能之差異。

關鍵詞：非線性 H_∞ 控制，能量耗散，Hamilton-Jacobi 方程式，倒單擺

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I. INTRODUCTION

The linear H_∞ control technique [1,2] has been one of the most prominent developments in the history of control theory and will continue to dominate robust control design for the years to come. Recently a much more complicated nonlinear H_∞ control has drawn attention to many investigators [3-7]. In [7], Ball, Helton, and Walker (BHW) successfully derived nonlinear H_∞ controller formulas which involve two Hamilton-Jacobi inequalities (HJIs). In order to obtain a nonlinear H_∞ controller based on the BHW's approach one needs to solve the HJIs or Hamilton-Jacobi equations (HJEs). Up to date, no computational algorithm is available for the exact explicit solution of HJE; however, an approximate solution can be obtained using successive computational methods [3,8-10]. In this paper, we will present a modified successive algorithm [11] which is developed based on Glad [9] and van der Shaft's [3] formulas. A numerical example is given to demonstrate the computational procedures.

Not many applications of the nonlinear H_∞ control are available in the literature. Part of the reason is the lack of efficient computation tool for the HJE. With the proposed algorithm for the HJE, we will present a nonlinear H_∞ controller design for the inverted pendulum control system and demonstrate the superiority of the nonlinear H_∞ controller to its linear counterpart. As shown in Fig. 1., the inverted pendulum system [12] is simply a stick mounted on a cart via a one-degree-of-freedom pivot. The inverted pendulum itself is an unstable nonlinear system. The objective is to design a (nonlinear H_∞) controller to drive the cart motor so that the cart can move back and forth maintaining the stick at a strictly vertical position. The inverted pendulum system will be formulated as a nonlinear H_∞ control problem. We will first solve the HJE and then use the solution to construct a nonlinear H_∞ controller. Simulations of the closed-loop system with the nonlinear H_∞ controller and its linear counterpart will be given and compared.

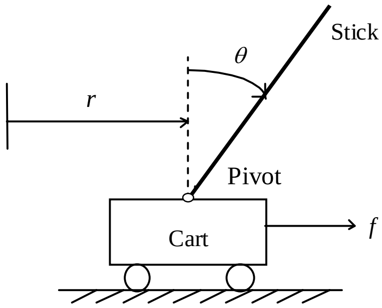

Fig.1. Inverted pendulum system.

Recently, Sheen [13] has found an exact solution of the nonlinear H_∞ control for the inverted pendulum system. In Sheen's work, some coefficients for the crucial function need to be carefully selected to make the function negative for all conditions; hence, the solution is neither unique nor optimal and is only applicable for the inverted pendulum problem. Although the performance of the controller obtained by the proposed algorithm is not as good as Sheen's method in this stage, due to only the third order approximate solution is applied, the solution from the proposed successive algorithm is unique, by solving the linear equations, and if one continuously proceeds the successive computation will obtain the optimal solution for the nonlinear H_∞ control of the inverted pendulum system. Besides, the proposed successive algorithm can be employed to some other practical control systems [14, 15].

The rest of the paper is organized as follows. In Section II, we briefly review the nonlinear H_∞ control problem, the nonlinear H_∞ controller formulas, and the related algebraic Riccati inequalities (ARIs). In Section III, a detailed successive computational procedure is proposed to find an approximate solution of the HJE. In Section IV, a nonlinear model of the inverted pendulum system is given and then a nonlinear H_∞ control problem is formulated in Section V. In section VI, we construct a nonlinear H_∞ controller and perform simulations of the closed-loop pendulum responses for the nonlinear H_∞ and the linear H_∞ controllers. Section VII gives the concluding remarks.

II. PRELIMINARIES

A. Notations

The notations used in this paper are fairly standard. $\mathbf{R}^n$ is the n-dimensional Euclidean space. $O(x^m)$ means the higher order terms including x^m. $(\cdot)^{(k)}$ stands for the k-th order term and $(\cdot)^{[k]}$ the sum of all terms up to the k-th order term. ARE is the algebraic Riccati equation and ARI is the algebraic Riccati inequality. HJE and HJI are the Hamilton-Jacobi equation and inequality, respectively. The notations regarding to the inverted pendulum system will be introduced in Section IV.

B. Nonlinear H_∞ Control Problem

Consider the following nonlinear input-affine generalized plant G:

$$G:\begin{cases}\dot{x}=f(x)+g_1(x)w+g_2(x)u\\ z=h_1(x)+D_{12}(x)u\\ y=h_2(x)+D_{21}(x)w\end{cases}\qquad(1)$$

where $x\in\mathbf{R}^n$ is the state of the system, $z\in\mathbf{R}^{p1}$ is the controlled output, $w\in\mathbf{R}^{m1}$ is the exogenous input including all commands and disturbances, $u\in\mathbf{R}^{m2}$ represents the control input, and $y\in\mathbf{R}^{p2}$ is the measured output, in which the standard description and the assumptions for the system can be referred to [3-7]. The problem is to find a controller

$$K:\begin{cases}\dot{\xi}=A_K(\xi)+B_K(\xi)y\\ u=C_K(\xi)\end{cases}\qquad(2)$$

so that the closed-loop system is locally asymptotically stable and γ-dissipative [14], or equivalently, the L_2-gain [3] of the system is less than or equal to a positive prescribed number γ.

C. Nonlinear H_∞ Controller Formulas

The nonlinear H_∞ controller formulas in [7], for convenience, are summarized in the following theorem.

Theorem 2.1 *Consider the nonlinear generalized plant defined in (1). If there exists a controller K of the form (2) such that the closed-loop system is locally asymptotically stable and γ-dissipative, then we have the following:*

(i) *There exist $X(x)$ and $Y_I(x)$ such that the following Hamilton-Jacobi inequalities:*

$$HJX(x) := 2X^T(x)H_A(x) + X^T(x)H_R(x)X(x) + H_Q(x) \le 0 \tag{3a}$$

$$HJY_I(x) := 2Y_I^T(x)J_A(x) + Y_I^T(x)J_R(x)Y_H(x) + J_Q(x) \le 0 \tag{3b}$$

are satisfied for all x in the domain of interest where

$$H_A(x) = f(x) - g_2(x)E_1^{-1}(x)D_{12}^T(x)h_1(x) \tag{4a}$$

$$H_R(x) = \gamma^{-2}g_1(x)g_1^T(x) - g_2(x)E_1^{-1}(x)g_2^T(x) \tag{4b}$$

$$H_Q(x) = h_1^T(x)h_1(x) - h_1^T(x)D_{12}(x)E_1^{-1}(x)D_{12}^T(x)h_1(x) \tag{4c}$$

$$J_A(x) = f(x) - g_1(x)D_{21}^T(x)E_2^{-1}(x)h_2(x) \tag{4d}$$

$$J_R(x) = \gamma^{-2}g_1(x)g_1^T(x) - \gamma^{-2}g_1(x)D_{21}^T(x)E_2^{-1}(x)D_{21}(x)g_1^T(x) \tag{4e}$$

$$J_Q(x) = h_1^T(x)h_1(x) - \gamma^2 h_2^T(x)E_2^{-1}(x)h_2(x) \tag{4f}$$

$$E_1(x) = D_{12}^T(x)D_{12}(x) \tag{4g}$$

$$E_2(x) = D_{21}(x)D_{21}^T(x) \tag{4h}$$

(ii) *$Y_I(x) - X(x)$ is the gradient of a positive function in the neighborhood of the equilibrium point.*

(iii) *A nonlinear H_∞ dissipative controller can be constructed by the following formulas:*

$$A_K(\xi) = f(\xi) + \gamma^{-2}[g_1(\xi) - B_K(\xi)D_{21}(\xi)]\cdot g_1^T(\xi)X(\xi) + g_2(\xi)C_K(\xi) - B_K(\xi)h_2(\xi) \tag{5a}$$

$$C_K(\xi) = -E_1^{-1}(\xi)[g_2^T(\xi)X(\xi) + D_{12}^T h_1(\xi)] \tag{5b}$$

where $B_K(\xi)$ satisfies the following equation:

$$[Y_I(\xi) - X(\xi)]^T B_K(\xi) = [\gamma^2 h_2^T(\xi) + Y_I^T(\xi)g_1(\xi)D_{21}^T(\xi))]E_2^{-1}(\xi_1) \tag{5c}$$

D. Linearized Model and ARIs

Assume the equilibrium point is at $x = \mathbf{0}$, the linearized model of the nonlinear generalized plant *G* in (2-1) can be obtained as follows:

$$G(s)_{linear} \quad : \begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{12} u \\ y = C_2 x + D_{21} w \end{cases} \tag{6}$$

and $E_1 = E_1(0),\ E_2 = E_2(0)$

To use a successive approximation algorithm for the solution of the HJIs, the first step is to solve their corresponding ARIs, i.e., to find $X > 0$ and $Y_I > 0$ so that the following three inequalities are satisfied.

$$RicX := (A - B_2E_1^{-1}D_{12}^TC_1)^T X + X(A - B_2E_1^{-1}D_{12}^TC_1) + X(\gamma^{-2}B_1B_1^T - B_2E_1^{-1}B_2^T)X + C_1^T(I - D_{12}E_1^{-1}D_{12}^T)C_1 \le 0 \tag{7a}$$

$$RicY_I := (A - B_1D_{21}^TE_2^{-1}C_2)^T Y_I + Y_I(A - B_1D_{21}^TE_2^{-1}C_2) + Y_I\gamma^{-2}B_1(I - D_{21}^TE_2^{-1}D_{21})B_1^TY_I + (C_1^TC_1 - \gamma^2C_2^TE_2^{-1}C_2) \le 0 \tag{7b}$$

$$Z := Y_I - X > 0 \tag{7c}$$

III. SUCCESSIVE ALGORITHM FOR SOLVING THE HJE

In the following, we will propose the successive algorithm and the computational procedures for solving the HJE. The HJE (3a) can be rewritten as:

$$V_x(x)H_A(x)+\frac{1}{4}V_x(x)H_R(x)V_x^T(x)+H_Q(x)=0 \tag{8a}$$

where

$$X(x)=\frac{1}{2}V_x^T(x) \tag{8b}$$

Define

$$A_s=A-B_2E_1^{-1}D_{12}^TC_1 \tag{9a}$$

$$R_s=\gamma^{-2}B_1B_1^T-B_2E_1^{-1}B_2^T \tag{9b}$$

$$Q_s=C_1^T(I-D_{12}E_1^{-1}D_{12}^T)C_1 \tag{9c}$$

$$F_c=A_s+R_sX \tag{9d}$$

where X is the positive semi-definite stabilizing solution of the ARE (7a). Let $f_h(x)=O(x^2)$, $R_h(x)=O(x)$, and $Q_h(x)=O(x^3)$ be the high-order terms that satisfy the following,

$$H_A(x)=A_sx+f_h(x) \tag{9e}$$

$$\frac{1}{4}H_R(x)=\frac{1}{4}R_s+R_h(x) \tag{9f}$$

$$H_Q(x)=x^TQ_sx+Q_h(x) \tag{9g}$$

The k-th order approximate solution of (8a) can be written as

$$V^{[k]}(x)=\sum_{m=2}^{k}V^{(m)}(x)=x^TXx+\sum_{m=3}^{k}V^{(m)}(x) \tag{10}$$

Using Eqs. (9), (10) and the ARE in (7a), we approximate (8a) to the following:

$$-\sum_{m=3}^{k}\frac{\partial V^{(m)}}{\partial x}F_cx=\frac{\partial V^{[k-1]}}{\partial x}f_h+\sum_{m=3}^{k-1}\frac{\partial V^{(m)}}{\partial x}\frac{1}{4}R_s\cdot\sum_{m=3}^{k-1}\frac{\partial^TV^{(m)}}{\partial x}+\frac{\partial V^{[k-1]}}{\partial x}R_h\frac{\partial^TV^{[k-1]}}{\partial x}+Q_h \tag{11}$$

The solutions of the HJE (3a) can be computed successively based on the following:

$$-\frac{\partial V^{(k)}}{\partial x}F_cx=\sum_{m=2}^{k-1}\frac{\partial V^{(m)}}{\partial x}f_h^{(k-m+1)}+\sum_{m=3}^{k-1}\frac{\partial V^{(k-m+2)}}{\partial x}\cdot\frac{1}{4}R_s\frac{\partial^TV^{(m)}}{\partial x}+\sum_{n=1}^{k-2}\sum_{m=2}^{k-n}\frac{\partial V^{(k-n-m+2)}}{\partial x}R_h^{(n)}\frac{\partial^TV^{(m)}}{\partial x}+Q_h^{(k)}:=H_m^{(k)}(x) \tag{12}$$

where $k\geq 3$ is an integer. By comparing the coefficients on both sides of Eq. (12), a set of linear equations are established and employed to solve $V^{(k)}$. Then, based on Eq. (8b), an approximate solution of the HJE in (3a), $X(x)$, is constructed as follows,

$$\begin{aligned}X^{[k-1]}(x)&=\frac{1}{2}\frac{\partial^TV^{[k]}}{\partial x}=Xx+\frac{1}{2}\sum_{m=3}^{k}\frac{\partial^TV^{(k)}}{\partial x}\\&=X^{[k-2]}(x)+\frac{1}{2}\frac{\partial^TV^{(k)}}{\partial x}\end{aligned} \tag{13}$$

A. Computational Procedure

1. The first-order approximate solution:

$$X^{[1]}(x)=\frac{1}{2}\frac{\partial^TV^{(2)}}{\partial x}=Xx \tag{14}$$

2. The second-order approximate solution:

Assume the number of possible third-order terms of x is n_3. Let $V^{(3)}(x)$ be the linear combination of these n_3 terms. For example, if $x=[x_1 \quad x_2 \quad x_3]^T$, then

$$\begin{aligned}V^{(3)}(x)&=c_1x_1^3+c_2x_2^3+c_3x_3^3+c_4x_1^2x_2\\&+c_5x_1^2x_3+c_6x_1x_2^2+c_7x_1x_3^2+c_8x_1x_2x_3\\&+c_9x_2^2x_3+c_{10}x_2x_3^2\end{aligned} \tag{15}$$

and $n_3=10$. Eq. (12) now becomes

$$-\frac{\partial V^{(3)}}{\partial x} F_c x = 2x^T X f_h^{(2)}(x) + 4x^T X R_h^{(1)}(x) X x + Q_h^{(3)}(x) \quad (16)$$

Note that both sides of (16) consist of only the third-order terms. By comparing the coefficients for both sides of (16), a set of n_3 linear equations are established to give the solution of $V^{(3)}(x)$. The second-order approximate solution of the HJE from (11) is

$$X^{[2]}(x) = \frac{1}{2}\frac{\partial^T V^{[3]}}{\partial x} = X^{[1]}(x) + \frac{1}{2}\frac{\partial^T V^{(3)}}{\partial x} = Xx + \frac{1}{2}\frac{\partial^T V^{(3)}}{\partial x} \quad (17)$$

3. The third-order approximate solution:

As before, assume $V^{(4)}(x)$ has n_4 forth-order terms of x. Equation (12) now is

$$-\frac{\partial V^{(4)}}{\partial x} F_c x = 2x^T X f_h^{(3)} + \frac{\partial V^{(3)}}{\partial x} f_h^{(2)} + \frac{\partial V^{(3)}}{\partial x}\frac{1}{4}(\gamma^{-2} B_1 B_1^T - B_2 B_2^T)\frac{\partial^T V^{(3)}}{\partial x} + 4\frac{\partial V^{(3)}}{\partial x} R_h^{(1)} X x + 4x^T X R_h^{(2)} X x + Q_h^{(4)} \quad (18)$$

$V^{(4)}(x)$ can be found as a solution of the n_4 linear equations obtained from comparing the coefficients of (18). The third-order approximate solution $X^{[3]}(x)$ is

$$X^{[3]}(x) = X^{[2]}(x) + \frac{1}{2}\frac{\partial^T V^{(4)}}{\partial x} \quad (19)$$

The successive computation procedure can continue to produce higher order approximations if higher accuracy is required.

IV. NONLINEAR MODEL FOR THE INVERTED PENDULUM

For the inverted pendulum shown in Fig.1., let r be the displacement of the pivot and θ the angular displacement of the stick. M and m are the mass of the cart and the stick respectively. L is the distance from the pivot to the center of gravity of the stick, g is the gravitational acceleration, J represents the moment of inertia with respect to the center of gravity of the stick, and f is the force applied to the cart. The dynamic equations are given as follows: [12]

$$(J + mL^2)\ddot{\theta} - mgL\sin\theta + mL\cos\theta\,\ddot{r} = 0 \quad (20a)$$

$$(M + m)\ddot{r} - mL\dot{\theta}^2\sin\theta + mL\ddot{\theta}\cos\theta = f \quad (20b)$$

For the ease of manipulation, the invertible nonlinear transformation [16] is employed to change the coordinates in the input space by defining a new input $u = \ddot{r}$ to the system. Since $J = \frac{1}{3}mL^2$, (20) can be simplified as:

$$\ddot{\theta} = \frac{3g}{4L}\sin\theta - \frac{3}{4L}\cos\theta \cdot u \quad (21a)$$

$$f = (M + m - 0.75m\cos^2\theta)u - mL\dot{\theta}^2\sin\theta + 0.75mg\cos\theta\sin\theta \quad (21b)$$

The motor output torque can be computed by

$$\tau = \frac{f}{r_g} \quad (22)$$

where r_g is the radius of the driving wheel. In practice, we assume that r and θ can be measured. Note that $\dot{\theta}$ can be computed via taking derivative to θ measured in order to provide sufficient information to f in (21b). The following data are used:

$$M = 0.455kg, m = 0.21kg, L = 0.3048m,$$

$$g = 9.807 \, m/\sec^2, \text{ and } r_g = 0.006349m$$

Define the state variables as:

$$x = [x_1 \quad x_2 \quad x_3 \quad x_4]^T = [r \quad \dot{r} \quad \theta \quad \dot{\theta}]^T \tag{23}$$

The nonlinear state equation can be written from (21a) as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \dot{r} \\ \ddot{r} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{r} \\ u \\ \dot{\theta} \\ \frac{3g}{4L}\sin\theta - \frac{3}{4L}\cos\theta \cdot u \end{bmatrix} \tag{24}$$

With the given data, (24) becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \\ x_4 \\ 24.1314x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -4.0219x_3^{\;3} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2.4606 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.2303x_3^{\;2} \end{bmatrix} u$$

(25a)

$$\begin{aligned} &:= f(x) + g_2(x)u \\ &:= Ax + f_h(x) + B_2u + g_{2h}(x)u \end{aligned}$$

Note that for simplicity, only the approximate terms less than third-order are listed. The measured outputs are r and θ, so the output equations can be represented by

$$y = \begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x := C_2x \tag{25b}$$

V. NONLINEAR H_∞ CONTROL PROBLEM FORMULATION

In this formulation, we assume that a disturbance d is injected into the system via the state equation and the measurement is contaminated by the noise n as shown in Fig. 2.

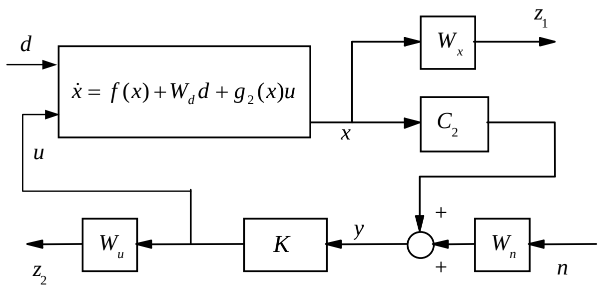

Fig.2. Nonlinear H_∞ control problem formulation for the inverted pendulum.

The weighted state vector, z_1, represents the disturbance response of interest to be minimized. The weighted control input, z_2, is employed to add control input constraint into the problem formulation. W_d, W_x, W_u, and W_n are appropriate constant weighting matrices. Let $w = [d \quad n]^T$ and $z = [z_1 \quad z_2]^T$, then the nonlinear generalized plant for the inverted pendulum control problem formulation is constructed as follows,

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)w + g_2(x)u \\ &= f(x) + [W_d \quad 0]w + g_2(x)u \\ &:= Ax + f_h(x) + B_1w + B_2u + g_{2h}(x)u \end{aligned} \tag{26a}$$

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = h_1(x) + D_{12}(x)u = \begin{bmatrix} W_x \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ W_u \end{bmatrix} u \tag{26b}$$

$$y = h_2(x) + D_{21}(x)w = C_2x + [0 \quad W_n]w \tag{26c}$$

The objective is to find a controller K such that the closed-loop system is stable and the H_∞ norm of the closed-loop transfer function from w to z, T_{zw}, is minimized. The signal z_1 in the controlled output $z=[z_1 \quad z_2]^T$ stands for the deviation of θ, r and their derivatives from the equilibrium, and the z_2 allows the control input constraint to be embedded in the problem formulation. The H_∞ norm of T_{zw} means the worst-case, i.e., the largest possible energy of z caused by the uncertain w with energy less than or equal to 1. The weighting matrices W_x, W_d, W_n, and W_u are chosen as:

$$W_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 10^{-6} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 10^{-6} \end{bmatrix}, W_d = I_4, W_n = I_2,$$

and $W_u = 1$ (27)

VI. NONLINEAR H_∞ CONTROL DESIGN AND SIMULATIONS

In this section, a nonlinear H_∞ controller for the nonlinear generalized plant of the inverted pendulum system given by (26) will be constructed.

The first step of the construction is to find the linear components of the solution. Consider the linearized model of (26). The optimal H_∞ norm of the linear closed-loop system is computed as γ_{opt} = 52.42 [1]. Choosing $\gamma = 55 > \gamma_{opt}$, one can find the solution for the ARE in (7a) as

$$X = \begin{bmatrix} 1.8258 & 1.6660 & 5.3303 & 1.0859 \\ 1.6660 & 2.4959 & 8.6347 & 1.7594 \\ 5.3303 & 8.6347 & 69.6382 & 14.1508 \\ 1.0859 & 1.7594 & 14.1508 & 2.8797 \end{bmatrix} \quad (28)$$

According to the successive algorithm presented in Section III, we have approximate solutions for the HJE (3a) in the following:
The first-order approximate solution is

$$X^{[1]}(x) = Xx \quad (29)$$

where X is given in (28). Due to $f_h^{(2)}(x) = R_h^{(1)}(x) = Q_h^{(3)}(x) = 0$ and from (16), we see that $V^{(3)}(x) = 0$. The second-order approximate solution is the same as the first-order one, i.e.,

$$X^{[2]}(x) = X^{[1]}(x) = Xx \quad (30)$$

That means there is no second-order term in this approximate solution. The construction of the third-order approximate solution is explained as follows. First, we compare the coefficients of the terms on both sides of (18) and set up 35 linear equations (since $n_4 = 35$), which in turn will be solved for $V^{(4)}(x)$ [11]. Then we have the third-order approximate solution $X^{[3]}(x)$ according to (19). Then, a nonlinear H_∞ controller can be obtained by plugging the third-order approximate solution of the HJE to the controller formulas (5).

A. Simulations

The computer simulations for the closed-loop system will be performed in the following to evaluate the performance of the proposed nonlinear H_∞ controller. The units for the system variables are $\theta : rad, r: m, u: m/\sec^2$, and $\tau : Nm$. Let the initial conditions be $[0 \quad 0 \quad \theta_0 \quad 0]^T$ and the exogenous input be

$$w = \begin{bmatrix} d \\ n \end{bmatrix} = A_w[1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]^T \tag{31}$$

where A_w is a constant representing the disturbance amplitude. The performance of the nonlinear H_∞ controller (based on the third-order approximate solution will be compared to that of the linear H_∞ controller [1].

First, the simulation is performed under the condition of no disturbance. The pendulum responses with $A_w = 0$ and small initial angle $\theta_0 = 0.2$ are plotted in Fig. 3. It shows that the pendulum angle θ converges to 0 after 3 seconds and the cart displacement r only deviates a little bit before it quickly returns to its equilibrium. Simulations for the linear H_∞ controller show that the performance of the linear controller is almost the same as that of the nonlinear controller at this point.

Now we increase the initial pendulum angular displacement to $\theta_0 = 0.5$ and let the disturbance amplitude be $A_w = 0.03$. The responses of θ and r for both the nonlinear H_∞ and linear H_∞ controllers are plotted in Fig. 4. and Fig. 5., respectively. The θ response for the nonlinear controller is faster, has less overshoot and smaller settling time. One observation from Fig. 5. is that less track length for the cart is needed for the nonlinear controller. The cart goes back to the origin and keeps the pendulum stable after 2.5 seconds for the nonlinear controller while the cart is continuously moving to maintain the pendulum vertically for the linear controller, which also reveals that the less input energy is consumed for nonlinear H_∞ controller. The reader can verify the fact easily by taking the second derivative of the cart moving displacement to obtain input u, defined as $u = \ddot{r}$ in Section III, and compare the area under the absolute value of the input plots.

The same simulation process is repeated with A_w increased to 0.15 and θ responses for both controllers are recorded in Fig. 6. It is obvious that the nonlinear H_∞ controller has better performance in keeping the stick straight. In the simulation we also found that an incredible 10m track is required for the linear controller while only 0.9m is needed for the nonlinear controller. If A_w is increased even larger to 0.2, then the θ response for linear H_∞ controller diverges while the response for the nonlinear H_∞ controller is able to converge to zero after only 4 seconds. On the other hand, if we increase the initial angle to $\theta_0 = 0.1$, the simulations show that the θ responses for the linear controller is divergent immediately even with no disturbance while the response for the nonlinear H_∞ controller is able to converge to zero after 7 seconds with $A_w = 1$.

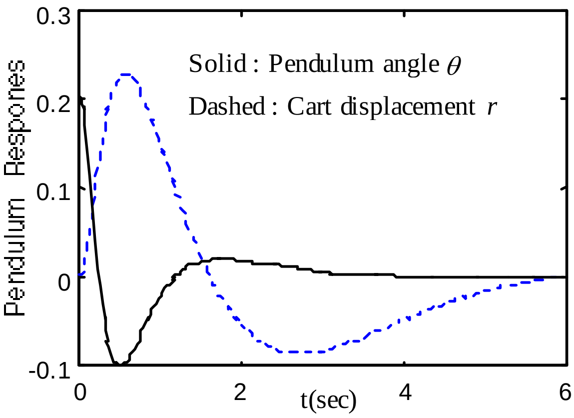

Fig.3. Pendulum responses with the nonlinear H_∞ controller when $\theta_0 = 0.2$ and $A_w = 0$.

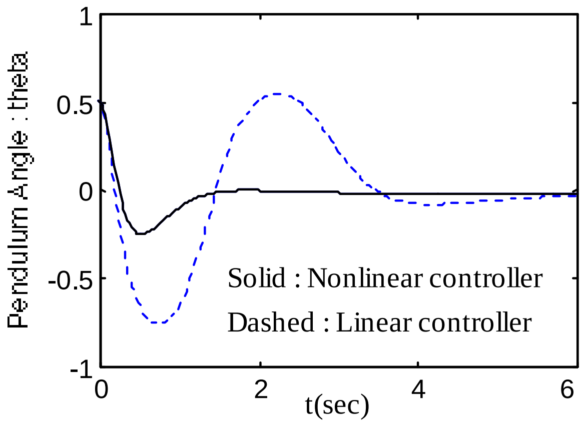

Fig.6. Comparison of pendulum angle θ responses when $\theta_0 = 0.5$ and $A_w = 0.15$.

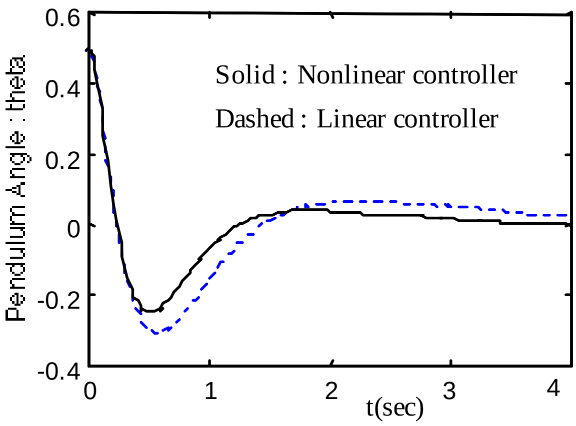

Fig.4. Comparison of the pendulum angle θ responses when $\theta_0 = 0.5$ and $A_w = 0.03$.

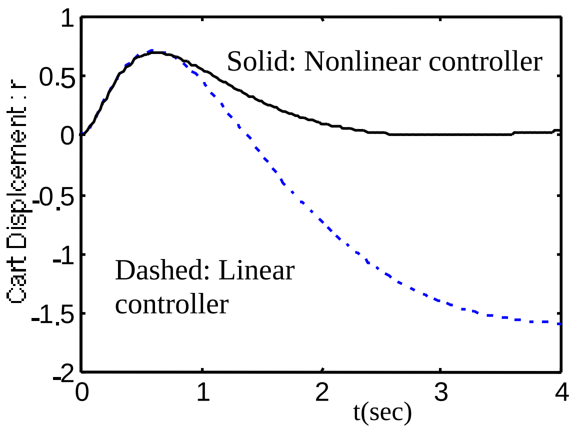

Fig.5. Comparison of the cart displacement r responses when $\theta_0 = 0.5$ and $A_w = 0.03$.

VII. CONCLUSIONS

In this paper, a successive algorithm was employed to construct a third-order approximate solution for the Hamilton-Jacobi equation. It is shown in the simulation for the closed-loop inverted pendulum system that the nonlinear H_∞ controller has much better performance and robustness than its linear counterpart. The time responses for the system with the nonlinear H_∞ controller converge back to the equilibrium much faster than those with the linear controller. The track length of the cart required to stabilize the inverted pendulum is much shorter for the nonlinear H_∞ controller.

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