中正嶺學報 第三十四卷 第一期 民國94.11.
JOURNAL OF C.C.I.T., VOL.34, NO.1, NOV., 2005

短文 Short Note

求解附帶任意個彈簧-質量懸吊系統之均勻矩形平板的自然頻率

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摘　　要

本文利用數值組合法來求解均勻矩形平板在任意位置附帶任意個數的彈簧-質量懸吊系統之自然頻率。首先由未附帶任何彈簧-質量懸吊系統的均勻矩形平板之運動方程式，吾人可得含有四個積分常數的特徵函數，將此特徵函數代入每一個彈簧-質量懸吊系統附著點的五個相容與平衡方程式中，則得一組聯立方程式。同樣的，將特徵函數分別代入均勻矩形平板之對稱 x 軸的左端與右端的邊界條件方程式中，吾人亦可分別得到另一組聯立方程式。將上述所有的聯立方程式寫成矩陣的形式，則得一特徵值方程式。因此一附帶 n 個彈簧-質量懸吊系統的均勻平板之總係數矩陣 $\left[\overline{B}\right]$ 之階數為 $5n+4$。求解 $\left|\overline{B}\right| = 0$ 可得有拘束的均勻矩形平板附帶 n 個彈簧-質量懸吊系統的自然頻率 $\omega_j\,(j = 1,2,\ldots)$。在現有文獻中，特徵方程式皆以數學展開顯示式之型式表示之，再利用解析法或數值法去求解。當一均勻矩形平板附帶二個(含)以上之彈簧-質量懸吊系統時，因其數學展開顯示式過於冗長而難以操作，故本文使用數值組合法以克服此一困難。

關鍵字：數值組合法，自然頻率，特徵方程式，拘束平板

The Solutions for the Natural Frequencies of Uniform Rectangular Plate Carrying Multiple Spring-Mass Systems

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ABSTRACT

From the motion equation of the "bare" uniform rectangular plate (without any spring-mass systems), an eigenfunction consisting of four integration constants is obtained. Where the last eigenfunction is substituted into the three compatible equations, one force-equilibrium equation and one governing equation for the v-th sprung mass (v=1,…,n) and the boundary equations for the two ends of the symmetry plate, a matrix equation of the form $\left[\overline{B}\right]\left\{\overline{C}\right\}=0$ is got. The solutions of $\left|\overline{B}\right| = 0$ (where $|\cdot|$ denotes a determinant) give the "exact" natural frequencies of the "constrained" plate (carrying any number of spring-mass systems). Since the order of $\left[\overline{B}\right]$ is $5n+4$, where n is the total number of spring-mass systems, the "explicit" mathematical expressions for the existing approach becomes lengthy intractable if $n > 2$. The "numerical assembly method" introduced in this paper aims at improving the last drawback of the existing approach.

Keywords: numerical assembly method, natural frequency, eigenfunction, constrained plate

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文稿收件日期 93.9.30; 文稿修正後以短文接受日期 94.2.14.
Manuscript received Sept. 30, 2004; revised as short note Feb. 14, 2005.

壹、前言

雖然用閉式解析解來求均勻矩形平板未受拘束(未攜帶任何懸吊系統)之自然頻率已不困難[1-3]。有關均勻矩形平板攜帶一彈性懸吊集中質量之振動問題，近來廣範被研究，1977 年 Laura 等[4]曾對一構件攜帶一彈性懸吊集中質量之動態行為研究。1993 年 Avalos 等[5]曾利用 Dirac delta 函數求取四周簡支之均勻矩形平板攜帶一彈簧-質量懸吊系統之自然頻率方程式並求其特徵值。1994 年 Avalos 等[6]利用傅立葉級數及貝色函數對一連續均勻圓形平板攜帶一彈簧-質量懸吊系統之自然頻率係數近似正解之探討。1997 年 Avalos 等[7]利用傅立葉級數對一連續四周簡支之均勻矩形平板攜帶一彈簧-質量懸吊系統之自然頻率係數閉式解之探討。至於利用頻率方程式展開各種邊界條件之均勻矩形平板攜帶一彈簧-質量懸吊系統之顯示式過於冗長，以至於在此方面研究之文獻不易尋獲，本文鑑於此亦為研究動機，因此而提出數值組合之技巧進行攜帶任意個數、大小、位置之彈簧-質量懸吊系統之均勻矩形平板特徵值分析，以克服現有之閉式解析法因顯示式過於冗長而不適用於求解一均勻矩形平板攜帶任意個的彈簧-質量懸吊系統的特徵值問題。

為了改進現存近似值方法，本文提出數值組合技巧來組合均勻平板攜帶任意個彈簧-質量懸吊系統的特徵方程式$\left[\overline{B}\right]\left\{\overline{C}\right\}=0$，然後利用傳統之半間距法求解均勻平板攜帶任意個彈簧-質量懸吊系統的特徵值的正解。本文提出數值組合技巧是將相關係數矩陣$\left[B_L\right]$、$\left[B_\upsilon\right]$和$\left[B_R\right]$，利用傳統的有限元素法之直接勁度矩陣法[8]加以組合，便可得到整體系統的頻率方程式$\left[\overline{B}\right]$。再利用傳統之半間距法求解均勻平板攜帶任意個彈簧-質量懸吊系統的頻率方程式之行列式值($\left|\overline{B}\right|=0$)，便可得到有 拘束平板之自然頻率 $\overline{\varpi}$ 或自然頻率係數。

為了確認本文提出之理論與方法之準確性，本文中分別計算在四個邊界條件下，均勻平板攜帶一個及五個彈簧-質量懸吊系統之前五個最低自然頻率。求得的前五個最低自然頻率分別與現有文獻或傳統的有限元素法(FEM)比較後，其自然頻率值或自然頻率係數值幾乎是完全一樣，因此本文理論與方法之可告性應可被接受。另本文所提到的四個邊界條件分別為：簡支-簡支-簡支-簡支(SSSS)，簡支-簡支-簡支-夾住(SSSC)，簡支-夾住-簡支-夾住(SCSC)和簡支-自由-簡支-自由(SFSF)。

貳、運動方程式和相容條件

若均勻矩形平板和第ν個懸吊彈簧-質量之運動方程式分別表示成[2,9]

$$D_E\nabla^4 w+\rho_A\ddot{w}=0 \tag{1}$$

及

$$m_\nu\ddot{z}_\nu+k_\nu(z_\nu-w_\nu)=0 \quad \text{或} \quad k_\nu(z_\nu-w_\nu)=-m_\nu\ddot{z}_\nu \tag{2}$$

均勻矩形平板在第ν個附著點需滿足下列的相容性(compatibility)

$$w_\nu^L(\xi_\nu,\eta_\nu,t)=w_\nu^R(\xi_\nu,\eta_\nu,t) \tag{3}$$

$$w_\nu'^L(\xi_\nu,\eta_\nu,t)=w_\nu'^R(\xi_\nu,\eta_\nu,t) \tag{4}$$

$$\frac{\partial^2 w_\nu^L(\xi_\nu,\eta_\nu,t)}{\partial\eta^2}+\upsilon\phi^2\frac{\partial^2 w_\nu^L(\xi_\nu,\eta_\nu,t)}{\partial\xi^2}= \frac{\partial^2 w_\nu^R(\xi_\nu,\eta_\nu,t)}{\partial\eta^2}+\upsilon\phi^2\frac{\partial^2 w_\nu^R(\xi_\nu,\eta_\nu,t)}{\partial\xi^2} \tag{5}$$

均勻矩形平板與懸吊質量間之力平衡需滿足[3,4]

$$D_E\left[\frac{\partial^3 w_\nu^L(\xi_\nu,\eta_\nu,t)}{\partial\eta^3}+(2-\upsilon)\phi^2\frac{\partial^3 w_\nu^L(\xi_\nu,\eta_\nu,t)}{\partial\xi^2\partial\eta}-\frac{\partial^3 w_\nu^R(\xi_\nu,\eta_\nu,t)}{\partial\eta^3}- (2-\upsilon)\phi^2\frac{\partial^3 w_\nu^R(\xi_\nu,\eta_\nu,t)}{\partial\xi^2\partial\eta}\right]=F_S \tag{6}$$

其中 $\xi_\nu = x_\nu / a \, , \eta_\nu = \mathrm{y}_\nu / b$

$\phi = b/a$ (aspect ratio)

$D_E = \dfrac{Eh^3}{12(1-\upsilon^2)}$ (flexural rigidity of plate)

$E =$ 楊氏模數 (Young’s modulus)

$h =$ 平板厚度

$\upsilon =$ 波森比(Poisson’s ratio)

$\rho_A =$ 平板單位面積之質量

$$F_S = (ab)\frac{\omega^2 m_\nu}{1-\omega^2 (m_\nu / k_\nu)} Z_\nu(\xi_\nu, \eta_\nu)$$

k_ν 及 m_ν 表示第 ν 個彈簧-質量懸吊系統之彈簧常數及點質量(point mass)，$\ddot{z}_\nu$ 及 z_ν 分別表示第 ν 個彈簧-質量懸吊系統(相對於其靜平衡位置)之加速度和位移，而上標 L 和 R 分別表示在第 ν 個附著點之左片段(segment)及右片段(segment)。

而四邊簡支撐(simply supported-simply supported-simply supported-simply supported)矩形板之邊界條件為

$$w_\nu^L(\xi,\eta,t) = \left.\frac{\partial^2 w_\nu^L(\xi,\eta,t)}{\partial \eta^2}\right|_{\eta=0} = 0 \qquad (7)$$

$$w_\nu^R(\xi,\eta,t) = \left.\frac{\partial^2 w_\nu^R(\xi,\eta,t)}{\partial \eta^2}\right|_{\eta=1} = 0 \qquad (8)$$

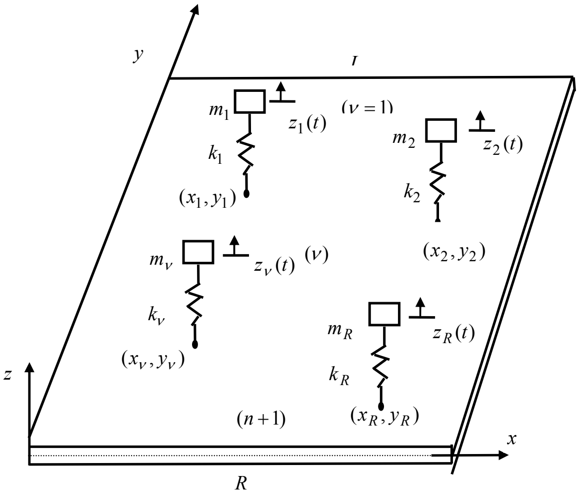

圖 1. 一均勻矩形平板附帶 n 個彈簧−質量懸吊系統。

參、頻率方程式之推導

當受約束的均勻平板做簡諧的自由振動時，均勻平板和懸吊質量之瞬時位移為

$$w(\xi,\eta,t) = W(\xi,\eta)e^{i\omega t} \qquad (9)$$

$$z_\nu(\xi,\eta,t) = Z_\nu(\xi,\eta)e^{i\omega t} \qquad (10)$$

將(9)式代入(1)式，可得自由振動之無因次控制方程式(dimensionless governing equation)為

$$\frac{\partial^4 W(\xi,\eta)}{\partial \eta^4} + 2\phi^2 \frac{\partial^4 W(\xi,\eta)}{\partial \eta^2 \partial \xi^2} + \phi^4 \frac{\partial^4 W(\xi,\eta)}{\partial \xi^4} - \phi^4 \lambda^2 W(\xi,\eta) = 0 \qquad (11)$$

其中

$$\lambda = \omega a^2 \sqrt{\rho / D_E}$$

根據 Levy’s Method，平板的垂向模態位移 $W(\xi,\eta)$ 及懸吊質量位移 $Z_\nu(\xi,\eta)$ 可以分別以下式表示之

$$W(\xi,\eta) = \sum_{m=1}^{\infty} Y_m(\eta) \sin m\pi\xi \qquad (12a)$$

$$Z_\nu(\xi,\eta) = \sum_{m=1}^{\infty} \hat{Z}_\nu(\eta) \sin m\pi\xi \qquad (12b)$$

將(12a)式代入(11)式得

$$\frac{d^4 Y_m(\eta)}{d\eta^4} - 2\phi^2 (m\pi)^2 \frac{d^2 Y_m(\eta)}{d\eta^2} + \phi^4\left[(m\pi)^4 - \lambda^2\right] Y_m(\eta) = 0 \;\; (m = 1,2,3,\cdots) \qquad (13)$$

吾人可解得(13)式之解析解可分成兩部分[2]

(I)當 $\lambda > (m\pi)^2$ 時

$$Y_m(\eta) = A_m \cosh\beta_1\eta + B_m \sinh\beta_1\eta + C_m \cos\beta_2\eta + D_m \sin\beta_2\eta \qquad (14)$$

對第 ν 個片段而言，(14)式可重寫成

$$Y_{m,\nu}(\eta_\nu) = A_{m,\nu} \cosh\beta_1\eta_\nu + B_{m,\nu} \sinh\beta_1\eta_\nu + C_{m,\nu} \cos\beta_2\eta_\nu + D_{m,\nu} \sin\beta_2\eta_\nu \qquad (15)$$

其中

$$\beta_1 = \phi\sqrt{\lambda \; + (m\pi)^2} \; , \; \beta_2 = \phi\sqrt{\lambda \; - (m\pi)^2}$$

由(2)式、(9)式、(10)式及(12)式吾人可得

$$k_\nu \sum_{m=1}^{\infty} Y_{m,\nu}(\eta_\nu)\sin m\pi\xi_\nu - (k_\nu - m_\nu\omega^2)\sum_{m=1}^{\infty} \hat{Z}_\nu(\eta_\nu)\sin m\pi\xi_\nu = 0 \tag{16a}$$

或

$$\sum_{m=1}^{\infty} Y_{m,\nu}(\eta_\nu)\sin m\pi\xi_\nu + (-1+\gamma_\nu^2)\sum_{m=1}^{\infty} \hat{Z}_\nu(\eta_\nu)\sin m\pi\xi_\nu = 0 \tag{16b}$$

其中

$$\gamma_\nu^2 = \frac{\omega^2}{\omega_\nu} = \frac{\lambda^2 D_E}{\rho a^4}\cdot\frac{m_\nu}{k_\nu} = \frac{\lambda^2 D_E}{\rho a^4}\cdot\frac{m_\nu^*\rho ab}{k_\nu^* D_E/a^2} = \phi\lambda^2\frac{m_\nu^*}{k_\nu^*}$$

$$m_\nu^* = \frac{m_\nu}{\rho ab} \text{ , } k_\nu^* = \frac{k_\nu a^2}{D_E}$$

將(16)式改寫成

$$\sum_{m=}^{\infty} Y_{m,\nu}(\eta_\nu)\sin m\pi\xi_\nu + (-1+\phi\lambda^2\frac{m_\nu^*}{k_\nu^*})\sum_{m=1}^{\infty} \hat{Z}_\nu(\eta_\nu)\sin m\pi\xi_\nu = 0 \tag{17}$$

將(15)式代入(17)式可得

$$\begin{aligned}&\sum_{m=1}^{\infty}(A_{m,\nu}\cosh\beta_1\eta_\nu + B_{m,\nu}\sinh\beta_1\eta_\nu + C_{m,\nu}\cos\beta_2\eta_\nu + D_{m,\nu}\sin\beta_2\eta_\nu)\sin m\pi\xi_\nu \\ &\quad + (-1+\phi\lambda^2\frac{m_\upsilon^*}{k_\nu^*})\sum_{m=1}^{\infty}\hat{Z}_\nu(\eta_\nu)\sin m\pi\xi_\nu = 0\end{aligned} \tag{18}$$

同樣地，將(9)式及(10)式代入(3)式~(6)式，吾人可得

$$W_{m,\nu}^L(\xi_\nu,\eta_\nu) = W_{m,\nu}^R(\xi_\nu,\eta_\nu) \tag{19}$$

$$W'^L_{m,\nu}(\xi_\nu,\eta_\nu) = W'^R_{m,\nu}(\xi_\nu,\eta_\nu) \tag{20}$$

$$\begin{aligned}&\frac{\partial^2 W_\nu^L(\xi_\nu,\eta_\nu)}{\partial\eta^2} + \upsilon\phi^2\frac{\partial^2 W_\nu^L(\xi_\nu,\eta_\nu)}{\partial\xi^2} = \\ &\qquad \frac{\partial^2 W_\nu^R(\xi_\nu,\eta_\nu)}{\partial\eta^2} + \upsilon\phi^2\frac{\partial^2 W_\nu^R(\xi_\nu,\eta_\nu)}{\partial\xi^2}\end{aligned} \tag{21}$$

$$\begin{aligned}&\left[\frac{\partial^3 W_\nu^L(\xi_\nu,\eta_\nu)}{\partial\eta^3} + (2-\upsilon)\phi^2\frac{\partial^3 W_\nu^L(\xi_\nu,\eta_\nu)}{\partial\xi^2\partial\eta} - \frac{\partial^3 W_\nu^R(\xi_\nu,\eta_\nu)}{\partial\eta^3} - \right. \\ &\qquad \left.(2-\upsilon)\phi^2\frac{\partial^3 W_\nu^R(\xi_\nu,\eta_\nu)}{\partial\xi^2\partial\eta}\right] - \Theta_\nu Z_\nu(\xi_\nu,\eta_\nu) = 0\end{aligned} \tag{22}$$

其中

$$\begin{aligned}\Theta_\nu &= \frac{(ab)}{D_E}\frac{\omega^2 m_\nu}{1-\omega^2\left(\frac{m_\nu}{k_\nu}\right)} = \frac{\frac{ab}{D_E}\frac{\lambda^2 D_E m_\nu}{\rho_A a^4}}{1-\frac{\lambda^2 D_E}{\rho_A a^4}\frac{m_\nu}{k_\nu}} \\ &= \frac{\lambda^2\left(\frac{m_\nu}{m_p}\right)\left(\frac{b}{a}\right)^2}{1-\lambda^2\left(\frac{m_\nu}{m_p}\right)\left(\frac{D_E}{k_\nu a^2}\right)\left(\frac{b}{a}\right)} = \frac{\lambda^2 m_\nu^*\phi^2}{1-\lambda^2\delta^2\phi}\end{aligned}$$

$$m_\nu^* = \frac{m_\nu}{m_p} = \frac{m_\nu}{\rho_A ab} \text{ ，} \delta^2 = \left(\frac{m_\nu}{m_p}\right)\left(\frac{D_E}{k_\nu a^2}\right)$$

將(9)式及(10)式代入(7)式~(8)式，吾人可得

$$W_\nu^L(\xi,\eta) = \left.\frac{\partial^2 W_\nu^L(\xi,\eta)}{\partial\eta^2}\right|_{\eta=0} = 0 \tag{23}$$

$$W_\nu^R(\xi,\eta) = \left.\frac{\partial^2 W_\nu^R(\xi,\eta)}{\partial\eta^2}\right|_{\eta=1} = 0 \tag{24}$$

由圖 1 吾人可知均勻板之左段片，L，與第一個部份($\nu=1$)之左段片重合，因此將(23)式和(12)式代入(15)式，吾人可得

$$A_{m,1} + C_{m,1} = 0 \tag{25}$$

$$A_{m,1}\beta_1^2 - C_{m,1}\beta_2^2 = 0 \tag{26}$$

將(25)式及(26)式改寫成矩陣型式

$$[B_L]\{C_L\} = 0 \tag{27}$$

其中

$$[B_L] = \begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ \hline 1 & 0 & 1 & 0 & 1 \\ \beta_1^2 & 0 & -\beta_2^2 & 0 & 2 \end{array} \tag{28}$$

$$\{C_L\} = \{A_{m,1}\ B_{m,1}\ C_{m,1}\ D_{m,1}\} = \{\overline{C}_1\ \overline{C}_2\ \overline{C}_3\ \overline{C}_4\} \tag{29}$$

而 $\overline{C}_1 = A_{m,1}$，$\overline{C}_2 = B_{m,1}$，$\overline{C}_3 = C_{m,1}$，$\overline{C}_4 = D_{m,1}$ (30)

在(28)式及後續的方程式中，在矩陣上邊及右邊的數目字表示與$\overline{C}_i(i=1,2,\cdots)$相關聯的自由度之辨識碼。

對第ν個附著點而言，由(19)~(22)式、(12)式及(15)式，吾人可得

$$
\begin{aligned}
&A_{m,\nu}\cosh\beta_1\eta_\nu + B_{m,\nu}\sinh\beta_1\eta_\nu + C_{m,\nu}\cos\beta_2\eta_\nu + D_{m,\nu}\sin\beta_2\eta_\nu \\
&- A_{m,\nu+1}\cosh\beta_1\eta_\nu - B_{m,\nu+1}\sinh\beta_1\eta_\nu - C_{m,\nu+1}\cos\beta_2\eta_\nu \\
&- D_{m,\nu+1}\sin\beta_2\eta = 0
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
&A_{m,\nu}\beta_1\sinh\beta_1\eta_\nu + B_{m,\nu}\beta_1\cosh\beta_1\eta_\nu - C_{m,\nu}\beta_2\sin\beta_2\eta_\nu \\
&+ D_{m,\nu}\beta_2\cos\beta_2\eta_\nu - A_{m,\nu+1}\beta_1\sinh\beta_1\eta_\nu \\
&- B_{m,\nu+1}\beta_1\cosh\beta_1\eta_\nu + C_{m,\nu+1}\beta_2\sin\beta_2\eta_\nu \\
&- D_{m,\nu+1}\beta_2\cos\beta_2\eta_\nu = 0
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
\sum_{m=1}^{\infty}\{&A_{m,\nu}\left[\beta_1^2 - \upsilon\phi^2(m\pi)^2\right]\cosh\beta_1\eta_\nu \\
&+ B_{m,\nu}\left[\beta_1^2 - \upsilon\phi^2(m\pi)^2\right]\sinh\beta_1\eta_\nu \\
&- C_{m,\nu}\left[\beta_2^2 + \upsilon\phi^2(m\pi)^2\right]\cos\beta_2\eta_\nu \\
&- D_{m,\nu}\left[\beta_2^2 + \upsilon\phi^2(m\pi)^2\right]\sin\beta_2\eta_\nu \\
&- A_{m,\nu+1}\left[\beta_1^2 - \upsilon\phi^2(m\pi)^2\right]\cosh\beta_1\eta_\nu \\
&- B_{m,\nu+1}\left[\beta_1^2 - \upsilon\phi^2(m\pi)^2\right]\sinh\beta_1\eta_\nu \\
&+ C_{m,\nu+1}\left[\beta_2^2 + \upsilon\phi^2(m\pi)^2\right]\cos\beta_2\eta_\nu \\
&+ D_{m,\nu+1}\left[\beta_2^2 + \upsilon\phi^2(m\pi)^2\right]\sin\beta_2\eta_\nu\} = 0
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
\sum_{m=1}^{\infty}\{&A_{m,\nu}\left[\beta_1^3 - (2-\upsilon)\phi^2(m\pi)^2\beta_1\right]\sinh\beta_1\eta_\nu\sin m\pi\xi_\nu \\
&+ B_{m,\nu}\left[\beta_1^3 - (2-\upsilon)\phi^2(m\pi)^2\beta_1\right]\cosh\beta_1\eta_\nu\sin m\pi\xi_\nu \\
&+ C_{m,\nu}\left[\beta_2^3 + (2-\upsilon)\phi^2(m\pi)^2\beta_2\right]\sin\beta_2\eta_\nu\sin m\pi\xi_\nu \\
&- D_{m,\nu}\left[\beta_2^3 + (2-\upsilon)\phi^2(m\pi)^2\beta_2\right]\cos\beta_2\eta_\nu\sin m\pi\xi_\nu \\
&- A_{m,\nu+1}\left[\beta_1^3 - (2-\upsilon)\phi^2(m\pi)^2\beta_1\right]\sinh\beta_1\eta_\nu\sin m\pi\xi_\nu \\
&- B_{m,\nu+1}\left[\beta_1^3 - (2-\upsilon)\phi^2(m\pi)^2\beta_1\right]\cosh\beta_1\eta_\nu\sin m\pi\xi_\nu \\
&- C_{m,\nu+1}\left[\beta_2^3 + (2-\upsilon)\phi^2(m\pi)^2\beta_2\right]\sin\beta_2\eta_\nu\sin m\pi\xi_\nu \\
&+ D_{m,\nu+1}\left[\beta_2^3 + (2-\upsilon)\phi^2(m\pi)^2\beta_2\right]\cos\beta_2\eta_\nu\sin m\pi\xi_\nu \\
&- \Theta_\nu\hat{Z}_\nu(\eta_\nu)\sin m\pi\xi_\nu\} = 0
\end{aligned}
\tag{34}
$$

值得注意的是，在(18)式及(31)~(34)式中，在第ν個附著點(ξ_ν,η_ν)處之左邊是屬於片段ν，而其右邊則是屬於片段$\nu+1$，因此，其相關聯係數分別以($A_{m,\nu}$ $B_{m,\nu}$ $C_{m,\nu}$ $D_{m,\nu}$)及($A_{m,\nu+1}$ $B_{m,\nu+1}$ $C_{m,\nu+1}$ $D_{m,\nu+1}$)表示之。

將(18)式及(31)~(34)式改寫成矩陣之型態，得

$$[B_\nu]\{C_\nu\} = 0 \tag{35}$$

其中

$$
\begin{array}{ccccccccc|c}
4\nu-3 & 4\nu-2 & 4\nu-1 & 4\nu & 4\nu+1 & 4\nu+2 & 4\nu+3 & 4\nu+4 & 4\nu+5 & \\
ch\beta_1\eta_\nu & sh\beta_1\eta_\nu & c\beta_2\eta_\nu & s\beta_2\eta_\nu & -ch\beta_1\eta_\nu & -sh\beta_1\eta_\nu & -c\beta_2\eta_\nu & -s\beta_2\eta_\nu & 0 & 5\nu-2 \\
\beta_1 sh\beta_1\eta_\nu & \beta_1 ch\beta_1\eta_\nu & -\beta_2 s\beta_2\eta_\nu & \beta_2 c\beta_2\eta_\nu & -\beta_1 sh\beta_1\eta_\nu & -\beta_1 ch\beta_1\eta_\nu & \beta_2 s\beta_2\eta_\nu & -\beta_2 c\beta_2\eta_\nu & 0 & 5\nu-1 \\
\Re ch\beta_1\eta_\nu & \Re sh\beta_1\eta_\nu & -\aleph c\beta_2\eta_\nu & -\aleph s\beta_2\eta_\nu & -\Re ch\beta_1\eta_\nu & -\Re sh\beta_1\eta_\nu & \aleph c\beta_2\eta_\nu & \aleph s\beta_2\eta_\nu & 0 & 5\nu \\
\kappa sh\beta_1\eta_\nu sm\pi\xi_\nu & \kappa ch\beta_1\eta_\nu sm\pi\xi_\nu & \chi s\beta_2\eta_\nu sm\pi\xi_\nu & -\chi c\beta_2\eta_\nu sm\pi\xi_\nu & -\kappa sh\beta_1\eta_\nu sm\pi\xi_\nu & -\kappa ch\beta_1\eta_\nu sm\pi\xi_\nu & -\chi s\beta_2\eta_\nu sm\pi\xi_\nu & \chi c\beta_2\eta_\nu sm\pi\xi_\nu & -\sum_{m=1}^{\infty}\Theta sm\pi\xi_\nu & 5\nu+1 \\
ch\beta_1\eta_\nu sm\pi\xi_\nu & sh\beta_1\eta_\nu sm\pi\xi_\nu & c\beta_2\eta_\nu sm\pi\xi_\nu & s\beta_2\eta_\nu sm\pi\xi_\nu & 0 & 0 & 0 & 0 & \Delta sm\pi\xi_\nu & 5\nu+2
\end{array}
\tag{36}
$$

$ch\beta_1\eta_\nu = \cosh\beta_1\eta_\nu$，$sh\beta_1\eta_\nu = \sinh\beta_1\eta_\nu$，$ch\beta_2\eta_\nu = \cosh\beta_2\eta_\nu$，$sh\beta_2\eta_\nu = \sinh\beta_2\eta_\nu$，$c\beta_1\eta_\nu = \cos\beta_1\eta_\nu$，$s\beta_1\eta_\nu = \sin\beta_1\eta_\nu$，$c\beta_2\eta_\nu = \cos\beta_2\eta_\nu$，$s\beta_2\eta_\nu = \sin\beta_2\eta_\nu$，$sm\pi\xi_\nu = \sin m\pi\xi_\nu$，$\Re = \sum_{m=1}^{\infty}\left[\beta_1^2 - \upsilon\phi^2(m\pi)^2\right]$，$\aleph = \sum_{m=1}^{\infty}\left[\beta_2^2 + \upsilon\phi^2(m\pi)^2\right]$，$\kappa = \sum_{m=1}^{\infty}\left[\beta_1^3 - (2-\upsilon)\phi^2(m\pi)^2\beta_1\right]$，$\chi = \sum_{m=1}^{\infty}\left[\beta_2^3 + (2-\upsilon)\phi^2(m\pi)^2\beta_2\right]$，$\Delta = \sum_{m=1}^{\infty} -1 + \phi\gamma^2\frac{m_\nu^*}{k_\nu^*}$。

$$\{C_\nu\} = \{A_{m,\nu}\ B_{m,\nu}\ C_{m,\nu}\ D_{m,\nu}\ A_{m,\nu+1}\ B_{m,\nu+1}\ C_{m,\nu+1}\ D_{m,\nu+1}\ \hat{Z}_\nu\}$$

$$\{\overline{C}_{4\nu-3}\ \overline{C}_{4\nu-2}\ \overline{C}_{4\nu-1}\ \overline{C}_{4\nu}\ \overline{C}_{4\nu+1}\ \overline{C}_{4\nu+2}\ \overline{C}_{4\nu+3}\ \overline{C}_{4\nu+4}\ \overline{C}_{4\nu+5}\} \tag{37}$$

且

$$\overline{C}_{4\nu-3} = A_{m,\nu}\text{，}\overline{C}_{4\nu-2} = B_{m,\nu}\text{，}\ldots\ \overline{C}_{4\nu+4} = D_{m,\nu+1}\text{，}\overline{C}_{4\nu+5} = \hat{Z}_\nu \tag{38}$$

由圖 1 可發現均勻板之右端，R，與第(n+1)個片段($\nu = n+1$)相重合，因此，將(24)式代入(12)式及(15)式中，可得

$$A_{m,n+1}\cosh\beta_1 + B_{m,n+1}\sinh\beta_1 + C_{m,n+1}\cos\beta_2 + D_{m,n+1}\sin\beta_2 = 0 \tag{39}$$

$$A_{m,n+1}\beta_1^2\cosh\beta_1 + B_{m,n+1}\beta_1^2\sinh\beta_1 - C_{m,n+1}\beta_2^2\cos\beta_2 - D_{m,n+1}\beta_2^2\sin\beta_2 = 0 \tag{40}$$

將(39)式及(40)式寫成矩陣型式，可得

$$[B_R]\{C_R\} = 0 \tag{41}$$

其中

$$[B_R] = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} 4\mathrm{n}+1 & 4\mathrm{n}+2 & 4\mathrm{n}+3 & 4\mathrm{n}+4 \\ \end{array}$$

$$[B_R] = \begin{array}{cccc|c} 4\mathrm{n}+1 & 4\mathrm{n}+2 & 4\mathrm{n}+3 & 4\mathrm{n}+4 & \\ \cosh\beta_1 & \sinh\beta_1 & \cos\beta_2 & \sin\beta_2 & p-1 \\ \beta_1^2\cosh\beta_1 & \beta_1^2\sinh\beta_1 & -\beta_2^2\cos\beta_2 & -\beta_2^2\sin\beta_2 & p \end{array} \tag{42}$$

$$\{C_R\} = \{A_{m,n+1}\ \ B_{m,n+1}\ \ C_{m,n+1}\ \ D_{m,n+1}\} = \{\overline{C}_{4n+1}\ \ \overline{C}_{4n+2}\ \ \overline{C}_{4n+3}\ \ \overline{C}_{4n+4}\} \tag{43}$$

$$\overline{C}_{4n+1} = A_{m,n+1}\text{ ，}\ \overline{C}_{4n+2} = B_{m,n+1}\text{ ，}\ \overline{C}_{4n+3} = C_{m,n+1}\text{ ，}\ \overline{C}_{4n+4} = D_{m,n+1} \tag{44}$$

(II) 當 $\lambda < (m\pi)^2$ 時

$$Y_m(\eta) = A_m\cosh\beta_1\eta + B_m\sinh\beta_1\eta + C_m\cosh\beta_2\eta + D_m\sinh\beta_2\eta \tag{45}$$

對第 ν 個片段而言，(45)式可重寫成

$$Y_{m,\nu}(\eta_\nu) = A_{m,\nu}\cosh\beta_1\eta_\nu + B_{m,\nu}\sinh\beta_1\eta_\nu + C_{m,\nu}\cosh\beta_2\eta_\nu + D_{m,\nu}\sinh\beta_2\eta_\nu \tag{46}$$

其中

$$\beta_1 = \phi\sqrt{(m\pi)^2 + \lambda}\ ,\ \beta_2 = \phi\sqrt{(m\pi)^2 - \lambda}$$

同理，吾人可依 $\lambda > (m\pi)^2$ 時之步驟推導出左、右邊界條件與連續條件方程式為

左邊界條件方程式：

$$A_{m,1} + C_{m,1} = 0 \tag{47}$$

$$A_{m,1}\beta_1^2 + C_{m,1}\beta_2^2 = 0 \tag{48}$$

將(47)式及(48)式改寫成矩陣型式

$$[B_L]\{C_L\} = 0 \tag{49}$$

其中

$$[B_L] = \begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ 1 & 0 & 1 & 0 & 1 \\ \beta_1^2 & 0 & \beta_2^2 & 0 & 2 \end{array} \tag{50}$$

$$\{C_L\} = \{A_{m,1}\ \ B_{m,1}\ \ C_{m,1}\ \ D_{m,1}\} = \{\overline{C}_1\ \ \overline{C}_2\ \ \overline{C}_3\ \ \overline{C}_4\} \tag{51}$$

而 $\overline{C}_1 = A_{m,1}$ ，$\overline{C}_2 = B_{m,1}$ ，$\overline{C}_3 = C_{m,1}$ ，$\overline{C}_4 = D_{m,1}$ (52)

右邊界條件方程式：

$$A_{m,n+1}\cosh\beta_1 + B_{m,n+1}\sinh\beta_1 + C_{m,n+1}\cosh\beta_2 + D_{m,n+1}\sinh\beta_2 = 0 \tag{53}$$

$$A_{m,n+1}\beta_1^2\cosh\beta_1 + B_{m,n+1}\beta_1^2\sinh\beta_1 + C_{m,n+1}\beta_2^2\cosh\beta_2 + D_{m,n+1}\beta_2^2\sinh\beta_2 = 0 \tag{54}$$

將(53)式及(54)式寫成矩陣型式，可得

$$[B_R]\{C_R\} = 0 \tag{55}$$

其中

$$[B_R] = \begin{array}{cccc|c} 4\mathrm{n}+1 & 4\mathrm{n}+2 & 4\mathrm{n}+3 & 4\mathrm{n}+4 & \\ \cosh\beta_1 & \sinh\beta_1 & \cos\beta_2 & \sin\beta_2 & p-1 \\ \beta_1^2\cosh\beta_1 & \beta_1^2\sinh\beta_1 & \beta_2^2\cosh\beta_2 & \beta_2^2\sinh\beta_2 & p \end{array} \tag{56}$$

$$\{C_R\} = \{A_{m,n+1}\ \ B_{m,n+1}\ \ C_{m,n+1}\ \ D_{m,n+1}\} = \{\overline{C}_{4n+1}\ \ \overline{C}_{4n+2}\ \ \overline{C}_{4n+3}\ \ \overline{C}_{4n+4}\} \tag{57}$$

$$\overline{C}_{4n+1} = A_{m,n+1}\text{ ，}\ \overline{C}_{4n+2} = B_{m,n+1}\text{ ，}\ \overline{C}_{4n+3} = C_{m,n+1}\text{ ，}\ \overline{C}_{4n+4} = D_{m,n+1} \tag{58}$$

連續條件方程式：

$$\sum_{m=1}^{\infty}(A_{m,\nu}\cosh\beta_1\eta_\nu + B_{m,\nu}\sinh\beta_1\eta_\nu + C_{m,\nu}\cosh\beta_2\eta_\nu + D_{m,\nu}\sinh\beta_2\eta_\nu)\sin m\pi\xi_\nu + (-1 + \phi\lambda^2\frac{m_\upsilon^*}{k_\nu^*})\sum_{m=1}^{\infty}\hat{Z}_\nu(\eta_\nu)\sin m\pi\xi_\nu = 0 \tag{59}$$

其中 $m_\nu^* = \dfrac{m_\nu}{\rho ab}$ ， $k_\nu^* = \dfrac{k_\nu a^2}{D_E}$

$$A_{m,\nu}\cosh\beta_1\eta_\nu + B_{m,\nu}\sinh\beta_1\eta_\nu + C_{m,\nu}\cosh\beta_2\eta_\nu + D_{m,\nu}\sinh\beta_2\eta_\nu - A_{m,\nu+1}\cosh\beta_1\eta_\nu - B_{m,\nu+1}\sinh\beta_1\eta_\nu - C_{m,\nu+1}\cosh\beta_2\eta_\nu - D_{m,\nu+1}\sinh\beta_2\eta = 0 \tag{60}$$

$$\begin{aligned}&A_{m,\nu}\beta_1\sinh\beta_1\eta_\nu+B_{m,\nu}\beta_1\cosh\beta_1\eta_\nu+C_{m,\nu}\beta_2\sinh\beta_2\eta_\nu\\&+D_{m,\nu}\beta_2\cosh\beta_2\eta_\nu-A_{m,\nu+1}\beta_1\sinh\beta_1\eta_\nu\\&-B_{m,\nu+1}\beta_1\cosh\beta_1\eta_\nu-C_{m,\nu+1}\beta_2\sinh\beta_2\eta_\nu\\&-D_{m,\nu+1}\beta_2\cosh\beta_2\eta_\nu=0\end{aligned}\qquad(61)$$

將(59)~(63)式改寫成矩陣之型態，得

$$[B_\nu]\{C_\nu\}=0 \qquad (64)$$

其中

$$\begin{array}{cccccccccc}4\nu-3 & 4\nu-2 & 4\nu-1 & 4\nu & 4\nu+1 & 4\nu+2 & 4\nu+3 & 4\nu+4 & 4\nu+5 & \\ \left[\begin{matrix} ch\beta_1\eta_\nu \\ \beta_1 sh\beta_1\eta_\nu \\ \mathfrak{R}ch\beta_1\eta_\nu \\ \kappa sh\beta_1\eta_\nu sm\pi\xi_\nu \\ ch\beta_1\eta_\nu sm\pi\xi_\nu \end{matrix}\right. & \begin{matrix} sh\beta_1\eta_\nu \\ \beta_1 ch\beta_1\eta_\nu \\ \mathfrak{R}sh\beta_1\eta_\nu \\ \kappa ch\beta_1\eta_\nu sm\pi\xi_\nu \\ sh\beta_1\eta_\nu sm\pi\xi_\nu \end{matrix} & \begin{matrix} ch\beta_2\eta_\nu \\ \beta_2 sh\beta_2\eta_\nu \\ \aleph' ch\beta_2\eta_\nu \\ \chi' sh\beta_2\eta_\nu sm\pi\xi_\nu \\ ch\beta_2\eta_\nu sm\pi\xi_\nu \end{matrix} & \begin{matrix} sh\beta_2\eta_\nu \\ \beta_2 ch\beta_2\eta_\nu \\ \aleph' sh\beta_2\eta_\nu \\ \chi' ch\beta_2\eta_\nu sm\pi\xi_\nu \\ sh\beta_2\eta_\nu sm\pi\xi_\nu \end{matrix} & \begin{matrix} -ch\beta_1\eta_\nu \\ -\beta_1 sh\beta_1\eta_\nu \\ -\mathfrak{R}ch\beta_1\eta_\nu \\ -\kappa sh\beta_1\eta_\nu sm\pi\xi_\nu \\ 0 \end{matrix} & \begin{matrix} -sh\beta_1\eta_\nu \\ -\beta_1 ch\beta_1\eta_\nu \\ -\mathfrak{R}sh\beta_1\eta_\nu \\ -\kappa ch\beta_1\eta_\nu sm\pi\xi_\nu \\ 0 \end{matrix} & \begin{matrix} -ch\beta_2\eta_\nu \\ -\beta_2 sh\beta_2\eta_\nu \\ -\aleph' ch\beta_2\eta_\nu \\ -\chi' sh\beta_2\eta_\nu sm\pi\xi_\nu \\ 0 \end{matrix} & \begin{matrix} -sh\beta_2\eta_\nu \\ -\beta_2 ch\beta_2\eta_\nu \\ -\aleph' sh\beta_2\eta_\nu \\ -\chi' ch\beta_2\eta_\nu sm\pi\xi_\nu \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ 0 \\ 0 \\ -\sum_{m=1}^{\infty}\Theta sm\pi\xi_\nu \\ \Delta sm\pi\xi_\nu \end{matrix}\right] & \begin{matrix} 5\nu-2 \\ 5\nu-1 \\ 5\nu \\ 5\nu+1 \\ 5\nu+2 \end{matrix}\end{array}$$

(65)

$$\begin{aligned}&\sum_{m=1}^{\infty}\{A_{m,\nu}\left[\beta_1^2-\upsilon\phi^2(m\pi)^2\right]\cosh\beta_1\eta_\nu+B_{m,\nu}\left[\beta_1^2-\upsilon\phi^2(m\pi)^2\right]\sinh\beta_1\eta_\nu\\&+C_{m,\nu}\left[\beta_2^2-\upsilon\phi^2(m\pi)^2\right]\cosh\beta_2\eta_\nu+D_{m,\nu}\left[\beta_2^2-\upsilon\phi^2(m\pi)^2\right]\sinh\beta_2\eta_\nu\\&-A_{m,\nu+1}\left[\beta_1^2-\upsilon\phi^2(m\pi)^2\right]\cosh\beta_1\eta_\nu\\&-B_{m,\nu+1}\left[\beta_1^2-\upsilon\phi^2(m\pi)^2\right]\sinh\beta_1\eta_\nu\\&-C_{m,\nu+1}\left[\beta_2^2-\upsilon\phi^2(m\pi)^2\right]\cosh\beta_2\eta_\nu\\&-D_{m,\nu+1}\left[\beta_2^2-\upsilon\phi^2(m\pi)^2\right]\sinh\beta_2\eta_\nu\}=0\end{aligned}\qquad(62)$$

$$\begin{aligned}&\sum_{m=1}^{\infty}\{A_{m,\nu}\left[\beta_1^3-(2-\upsilon)\phi^2(m\pi)^2\beta_1\right]\sinh\beta_1\eta_\nu\sin m\pi\xi_\nu\\&+B_{m,\nu}\left[\beta_1^3-(2-\upsilon)\phi^2(m\pi)^2\beta_1\right]\cosh\beta_1\eta_\nu\sin m\pi\xi_\nu\\&+C_{m,\nu}\left[\beta_2^3-(2-\upsilon)\phi^2(m\pi)^2\beta_2\right]\sinh\beta_2\eta_\nu\sin m\pi\xi_\nu\\&+D_{m,\nu}\left[\beta_2^3-(2-\upsilon)\phi^2(m\pi)^2\beta_2\right]\cosh\beta_2\eta_\nu\sin m\pi\xi_\nu\\&-A_{m,\nu+1}\left[\beta_1^3-(2-\upsilon)\phi^2(m\pi)^2\beta_1\right]\sinh\beta_1\eta_\nu\sin m\pi\xi_\nu\\&-B_{m,\nu+1}\left[\beta_1^3-(2-\upsilon)\phi^2(m\pi)^2\beta_1\right]\cosh\beta_1\eta_\nu\sin m\pi\xi_\nu\\&-C_{m,\nu+1}\left[\beta_2^3-(2-\upsilon)\phi^2(m\pi)^2\beta_2\right]\sinh\beta_2\eta_\nu\sin m\pi\xi_\nu\\&-D_{m,\nu+1}\left[\beta_2^3(2-\upsilon)\phi^2(m\pi)^2\beta_2\right]\cosh\beta_2\eta_\nu\sin m\pi\xi_\nu\\&-\Theta_\nu\hat{Z}_\nu(\eta_\nu)\sin m\pi\xi_\nu\}=0\end{aligned}$$

(63)

其中

$$\Theta_\nu=\frac{(ab)}{D_E}\frac{\omega^2 m_\nu}{1-\omega^2\left(\frac{m_\nu}{k_\nu}\right)}=\frac{\frac{ab}{D_E}\frac{\lambda^2 D_E m_\nu}{\rho_A a^4}}{1-\frac{\lambda^2 D_E}{\rho_A a^4}\frac{m_\nu}{k_\nu}}$$

$$=\frac{\lambda^2\left(\frac{m_\nu}{m_p}\right)\left(\frac{b}{a}\right)^2}{1-\lambda^2\left(\frac{m_\nu}{m_p}\right)\left(\frac{D_E}{k_\nu a^2}\right)\left(\frac{b}{a}\right)}=\frac{\lambda^2 m_\nu^*\phi^2}{1-\lambda^2\delta^2\phi}$$

$$m_\nu^*=\frac{m_\nu}{m_p}=\frac{m_\nu}{\rho_A ab}\quad，\quad\delta^2=\left(\frac{m_\nu}{m_p}\right)\left(\frac{D_E}{k_\nu a^2}\right)$$

$ch\beta_1\eta_\nu=\cosh\beta_1\eta_\nu$ ， $sh\beta_1\eta_\nu=\sinh\beta_1\eta_\nu$ ，
$ch\beta_2\eta_\nu=\cosh\beta_2\eta_\nu$ ， $sh\beta_2\eta_\nu=\sinh\beta_2\eta_\nu$ ， $sm\pi\xi_\nu=\sin m\pi\xi_\nu$ ，

$$\mathfrak{R}=\sum_{m=1}^{\infty}\left[\beta_1^2-\upsilon\phi^2(m\pi)^2\right]\quad，\quad\aleph'=\sum_{m=1}^{\infty}\left[\beta_2^2-\upsilon\phi^2(m\pi)^2\right]\quad，$$

$$\kappa=\sum_{m=1}^{\infty}\left[\beta_1^3-(2-\upsilon)\phi^2(m\pi)^2\beta_1\right]\quad，$$

$$\chi'=\sum_{m=1}^{\infty}\left[\beta_2^3-(2-\upsilon)\phi^2(m\pi)^2\beta_2\right]\ ，\ \Delta=\sum_{m=1}^{\infty}-1+\phi\gamma^2\frac{m_\nu^*}{k_\nu^*}\ 。$$

$$\begin{aligned}\{C_\nu\}&=\{A_{m,\nu}\ B_{m,\nu}\ C_{m,\nu}\ D_{m,\nu}\ A_{m,\nu+1}\ B_{m,\nu+1}\ C_{m,\nu+1}\ D_{m,\nu+1}\ \hat{Z}_\nu\}\\&=\{\overline{C}_{4\nu-3}\ \overline{C}_{4\nu-2}\ \overline{C}_{4\nu-1}\ \overline{C}_{4\nu}\ \overline{C}_{4\nu+1}\ \overline{C}_{4\nu+2}\ \overline{C}_{4\nu+3}\ \overline{C}_{4\nu+4}\ \overline{C}_{4\nu+5}\}\end{aligned}\qquad(66)$$

且

$$\overline{C}_{4\nu-3}=A_{m,\nu}\ ，\ \overline{C}_{4\nu-2}=B_{m,\nu}\ ，\ \ldots\ \overline{C}_{4\nu+4}=D_{m,\nu+1}\ ，\ \overline{C}_{4\nu+5}=\hat{Z}_\nu\qquad(67)$$

$$p=5n+4\qquad(68)$$

上式中之 p 代表方程式之總數目。由以上的推導，吾人可知對一懸吊系統而言，每一附著點皆可得到五個方程式，且從每一邊界條件，吾人可得到二個方程式。因此，對於一附帶 n 個懸吊系統之均勻矩形板而言，所得到積分常數 $A_{m,\nu}$ 、 $B_{m,\nu}$ 、 $C_{m,\nu}$ 、 $D_{m,\nu}$ 及被懸吊質量之模態位移 $\hat{Z}_\nu$（$\nu=1,2,\cdots,n$）的方程式總數目為 $5n+4$，此即(68)式所示 $p=5n+4$。

若將所有的未知數 $A_{m,\nu}$ 、 $B_{m,\nu}$ 、 $C_{m,\nu}$ 、 $D_{m,\nu}$ 及 $\hat{Z}_\nu$ ($\nu=1,2,\cdots,n$)，以一係數 $\overline{C}_j(j=1,2,\cdots,p)$ [其定義如(29)、(37)、(43)、(51)、(57) 及(66)式所示]所組成的行向量 $\{\overline{C}\}$ 來表示，則矩陣 $[B_L]$ 、 $[B_\nu]$ 及 $[B_R]$ 與有限元素中之元素性質矩陣相似，其相對應的自由度之辨識碼，如(28)、(36)、(42)、(51)、(57) 及(66)式之上邊及右邊的數目計算式所示。利用直接勁度矩陣法之組合技巧，吾人便可得到整個振動系統之係數方程式為

$$[\overline{B}]\{\overline{C}\}=0 \tag{69}$$

若上述問題欲得非零解，則必需滿足

$$|\overline{B}|=0 \tag{70}$$

上式為一頻率方程式，本文利用文獻[10]的半間距數值法來求特徵值係數 λ_i 或特徵值 ω_i $(i=1,2,\cdots)$ 。

肆、決定各種支撐條件之 $[B_L]$ 及 $[B_R]$

由上一節吾人可知對於系統的附著點之 $[B_\nu]$ 與受拘束的均勻矩形平板之邊界條件無關，僅邊界矩陣 $[B_L]$ 及 $[B_R]$ 會因均勻矩形平板之邊界條件不同而有所不同，故必需根據不同的邊界條件加以修正，然後再按照本文所提之組合步驟來求解各種不同的邊界條件下受拘束的均勻矩形平板之特徵值分析。此乃本文方法最主要之優點。本文所考慮各種邊界條件下之邊界矩陣 $[B_L]$ 及 $[B_R]$ 如下：

當 $\lambda>(m\pi)^2$ 時

(1) 簡支-簡支-簡支-簡支矩形板(Simply supported- Simply supported- Simply supported- Simply supported plate)

$$[B_L]=\begin{array}{cc} \begin{array}{cccc}1&2&3&4\end{array} & \\ \left[\begin{array}{cccc}1&0&1&0\\ \beta_1^2&0&-\beta_2^2&0\end{array}\right] & \begin{array}{c}1\\2\end{array}\end{array} \tag{71}$$

$$[B_R]=\begin{array}{cc} \begin{array}{cccc}4n+1&4n+2&4n+3&4n+4\end{array} & \\ \left[\begin{array}{cccc}\cosh\beta_1&\sinh\beta_1&\cos\beta_2&\sin\beta_2\\ \beta_1^2\cosh\beta_1&\beta_1^2\sinh\beta_1&-\beta_2^2\cos\beta_2&-\beta_2^2\sin\beta_2\end{array}\right] & \begin{array}{c}p-1\\p\end{array}\end{array} \tag{72}$$

(2) 簡支-簡支-簡支-夾住矩形板(Simply supported- Simply supported- Simply supported- Clamped plate)

$$[B_L]=\begin{array}{cc} \begin{array}{cccc}1&2&3&4\end{array} & \\ \left[\begin{array}{cccc}1&0&1&0\\ \beta_1^2&0&-\beta_2^2&0\end{array}\right] & \begin{array}{c}1\\2\end{array}\end{array} \tag{73}$$

$$[B_R]=\begin{array}{cc} \begin{array}{cccc}4n+1&4n+2&4n+3&4n+4\end{array} & \\ \left[\begin{array}{cccc}\cosh\beta_1&\sinh\beta_1&\cos\beta_2&\sin\beta_2\\ \beta_1\sinh\beta_1&\beta_1\cosh\beta_1&-\beta_2\sin\beta_2&\beta_2\cos\beta_2\end{array}\right] & \begin{array}{c}p-1\\p\end{array}\end{array} \tag{74}$$

(3) 簡支-夾住-簡支-夾住矩形板(Simply supported-Clamped- Simply supported-Clamped plate)

$$[B_L]=\begin{array}{cc} \begin{array}{cccc}1&2&3&4\end{array} & \\ \left[\begin{array}{cccc}1&0&1&0\\ 0&\beta_1&0&\beta_2\end{array}\right] & \begin{array}{c}1\\2\end{array}\end{array} \tag{75}$$

$$[B_R]=\begin{array}{cc} \begin{array}{cccc}4n+1&4n+2&4n+3&4n+4\end{array} & \\ \left[\begin{array}{cccc}\cosh\beta_1&\sinh\beta_1&\cos\beta_2&\sin\beta_2\\ \beta_1\sinh\beta_1&\beta_1\cosh\beta_1&-\beta_2\sin\beta_2&\beta_2\cos\beta_2\end{array}\right] & \begin{array}{c}p-1\\p\end{array}\end{array} \tag{76}$$

(4) 簡支-自由-簡支-自由矩形板(Simply supported-Free-Simply supported- Free plate)

$$[B_L]=\begin{array}{cc} \begin{array}{cccc}1&2&3&4\end{array} & \\ \left[\begin{array}{cccc}\Re&0&\aleph&0\\ 0&\kappa&0&\chi\end{array}\right] & \begin{array}{c}1\\2\end{array}\end{array} \tag{77}$$

$$[B_R]=\begin{array}{cc} \begin{array}{cccc}4n+1&4n+2&4n+3&4n+4\end{array} & \\ \left[\begin{array}{cccc}\Re\cosh\beta_1&\Re\sinh\beta_1&-\aleph\cos\beta_2&-\aleph\sin\beta_2\\ \kappa\sinh\beta_1&\kappa\cosh\beta_1&\chi\beta_2\sin\beta_2&-\chi\beta_2\cos\beta_2\end{array}\right] & \begin{array}{c}p-1\\p\end{array}\end{array} \tag{78}$$

$$\Re=\sum_{m=1}^{\infty}\left[\beta_1^2-\upsilon\phi^2(m\pi)^2\right] \quad , \quad \aleph=\sum_{m=1}^{\infty}\left[\beta_2^2+\upsilon\phi^2(m\pi)^2\right] \quad ,$$

$$\kappa=\sum_{m=1}^{\infty}\left[\beta_1^3-(2-\upsilon)\phi^2(m\pi)^2\beta_1\right] \quad ,$$

$$\chi=\sum_{m=1}^{\infty}\left[\beta_2^3+(2-\upsilon)\phi^2(m\pi)^2\beta_2\right]$$

當 $\lambda<(m\pi)^2$ 時

(1) 簡支-簡支-簡支-簡支矩形板(Simply supported- Simply supported- Simply supported- Simply supported plate)

$$[B_L]=\begin{array}{cccccc} & 1 & 2 & 3 & 4 & \\ \big[& 1 & 0 & 1 & 0 & \big]\,1 \\ \big[& \beta_1^2 & 0 & \beta_2^2 & 0 & \big]\,2 \end{array} \qquad (79)$$

$$[B_R]=\begin{array}{cccccc} & 4n+1 & 4n+2 & 4n+3 & 4n+4 & \\ \big[& \cosh\beta_1 & \sinh\beta_1 & \cosh\beta_2 & \sinh\beta_2 & \big]\,p-1 \\ \big[& \beta_1^2\cosh\beta_1 & \beta_1^2\sinh\beta_1 & \beta_2^2\cosh\beta_2 & \beta_2^2\sinh\beta_2 & \big]\,p \end{array} \qquad (80)$$

(2) 簡支-簡支-簡支-夾住矩形板(Simply supported- Simply supported- Simply supported- Clamped plate)

$$[B_L]=\begin{array}{cccccc} & 1 & 2 & 3 & 4 & \\ \big[& 1 & 0 & 1 & 0 & \big]\,1 \\ \big[& \beta_1^2 & 0 & \beta_2^2 & 0 & \big]\,2 \end{array} \qquad (81)$$

$$[B_R]=\begin{array}{cccccc} & 4n+1 & 4n+2 & 4n+3 & 4n+4 & \\ \big[& \cosh\beta_1 & \sinh\beta_1 & \cosh\beta_2 & \sinh\beta_2 & \big]\,p-1 \\ \big[& \beta_1\sinh\beta_1 & \beta_1\cosh\beta_1 & \beta_2\sinh\beta_2 & \beta_2\cosh\beta_2 & \big]\,p \end{array} \qquad (82)$$

(3) 簡支-夾住-簡支-夾住矩形板(Simply supported-Clamped- Simply supported-Clamped plate)

$$[B_L]=\begin{array}{cccccc} & 1 & 2 & 3 & 4 & \\ \big[& 1 & 0 & 1 & 0 & \big]\,1 \\ \big[& 0 & \beta_1 & 0 & \beta_2 & \big]\,2 \end{array} \qquad (83)$$

$$[B_R]=\begin{array}{cccccc} & 4n+1 & 4n+2 & 4n+3 & 4n+4 & \\ \big[& \cosh\beta_1 & \sinh\beta_1 & \cosh\beta_2 & \sinh\beta_2 & \big]\,p-1 \\ \big[& \beta_1\sinh\beta_1 & \beta_1\cosh\beta_1 & \beta_2\sinh\beta_2 & \beta_2\cosh\beta_2 & \big]\,p \end{array} \qquad (84)$$

(4) 簡支-自由-簡支-自由矩形板(Simply supported-Free-Simply supported- Free plate)

$$[B_L]=\begin{array}{cccccc} & 1 & 2 & 3 & 4 & \\ \big[& \Re & 0 & \aleph' & 0 & \big]\,1 \\ \big[& 0 & \kappa & 0 & \chi' & \big]\,2 \end{array} \qquad (85)$$

$$[B_R]=\begin{array}{cccccc} & 4n+1 & 4n+2 & 4n+3 & 4n+4 & \\ \big[& \Re\cosh\beta_1 & \Re\sinh\beta_1 & \aleph'\cosh\beta_2 & \aleph'\sinh\beta_2 & \big]\,p-1 \\ \big[& \kappa\sinh\beta_1 & \kappa\cosh\beta_1 & \chi'\beta_2\sinh\beta_2 & \chi'\beta_2\cosh\beta_2 & \big]\,p \end{array} \qquad (86)$$

$$\Re=\sum_{m=1}^{\infty}\left[\beta_1^2-\upsilon\phi^2(m\pi)^2\right] \text{，} \quad \aleph'=\sum_{m=1}^{\infty}\left[\beta_2^2-\upsilon\phi^2(m\pi)^2\right] \text{，}$$

$$\kappa=\sum_{m=1}^{\infty}\left[\beta_1^3-(2-\upsilon)\phi^2(m\pi)^2\beta_1\right] \text{，}$$

$$\chi'=\sum_{m=1}^{\infty}\left[\beta_2^3-(2-\upsilon)\phi^2(m\pi)^2\beta_2\right]$$

伍、數值分析結果與討論

本文所探討的均勻矩形平板之尺寸及物理性質：板長 $a=2.0\,m$，板寬 $b=3.0\,m$，板厚 $h=0.005\,m$，柏森比 $\upsilon=0.3$，彎曲剛度 $D_E=Eh^3/[12(1-\nu^2)]=2.3478\times10^3$ $N\text{-}m$，質量密度 $\rho=7850\,kg/m^3$，板單位面積之質量 $\rho_A=\rho\,h=39.25\,kg/m^2$，楊氏模數 $E=2.051\times10^{11}\,N/m^2$，板之總質量 $m_p=\rho hab=235.5$ kg 和板之參考勁度 $k_p=D_E/a^2=5.8695\times10^2\,N/m$.。

為了方便起見，本文引用兩個參數：勁度參數 k_v^* 及質量參數 m_v^*，其定義如下：

$$k_v^*=k_v/k_p \quad \text{及} \quad m_v^*=m_v/m_p \text{，} \quad v=1,2,\cdots$$

其中 k_v 表懸吊系統之彈簧常數，而 m_v 表懸吊系統之質量。

為了方便起見，本文以下將分別以 C、F 及 S 來表示邊界(支撐)情況係夾住的、自由的及簡支的。因為每一塊平板均有四個邊，故本文以 C、F 及 S 等字母所組成的四字母字語(four-letter acronym)來描述一均勻矩形平板之四邊支撐情況，並以頭字語之第一字母來表示一均勻矩形平板左邊的支撐情況，然後循反時針方向，以頭字語之第二、三及四個字母來表示一均勻矩形平板之下邊、右邊及上邊的支撐

情況。對本文所探討的均勻矩形平板之四種支撐情況(如圖 2)而言，根據上述的定義，則可以 SSSS、SSSC、SCSC 及 SFSF 來描述。

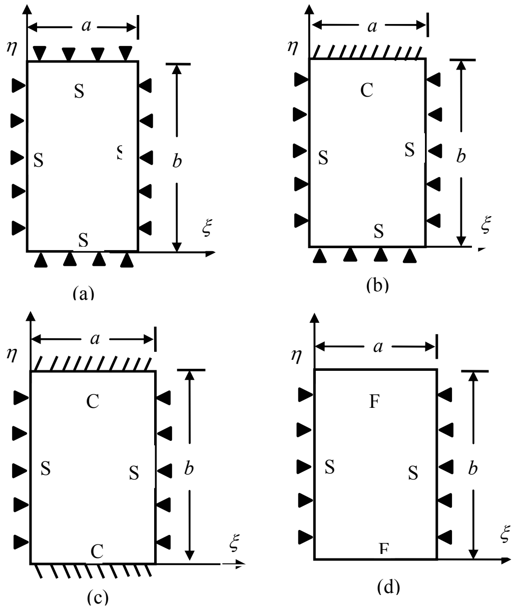

圖 2. 四種不同邊界條件之矩形平板: (a)SSSS; (b)SSSC; (c)SCSC; (d) SFSF。

5.1 理論及電算程式之可靠性

吾人將現有文獻[5]所舉的 SSSS 均勻矩形平板攜帶單一懸吊系統之實例，利用本文所提之方法(數值組合法)求解其特徵值並與文獻所得之結果比較。表 1、表 2 所示係一均勻四周簡支之矩形平板攜帶單一懸吊系統之前五個最低的特徵值係數 λ_i ($i=1,2,\cdots,5$) 及特徵值 ω_i ($i=1,2,\cdots,5$)(rad/sec)。由表 1、表 2 吾人發現文所提之方法(數值組合法)所求得之前五個最低的特徵值係數 λ_i ($i=1,2,\cdots,5$) 及特徵值 ω_i ($i=1,2,\cdots,5$)(rad/sec)與文獻[5]結果均非常接近。因此，這顯示本文的理論及所設計的電腦程式之可靠性應可被接受。值得注意的是文獻[5]中所求得之特徵值係數 λ_i ，該特徵值係數 λ_i 與表 2 的特徵值 ω_i 之間的關係為 $\omega_i^2 = \lambda_i^2 D_E / \rho h a^4$ ($i=1,2,\cdots,5$)。

表 1. 均勻矩形平板在 SSSS 邊界下附帶一個彈簧-質量系統所得之最低五個特徵值係數(懸吊系統位於 $\xi_1 = x_1/a = 0.5$, $\eta_1 = y_1/b = 0.5$ 和 b/a=0.5)。

k_1^*	m_1^*	Methods	Eigenvalue coefficients				
			λ_1	λ_2	λ_3	λ_4	λ_5
0.2	0.1	present	1.99999	49.34802	128.30486	365.17536	444.13220
		Ref.[5]	1.99923	49.36426	128.31109	365.17755	444.13400
	1.0	present	0.63246	49.34802	128.30486	365.17536	444.13220
		Ref.[5]	0.63221	49.36423	128.31109	365.17755	444.13400
	5.0	present	0.28284	49.34802	128.30486	365.17536	444.13220
		Ref.[5]	0.28273	49.36423	128.31109	365.17755	444.13400

Note： $\lambda_i^2 = \omega_i^2 \rho h a^4 / D_E \left(i=1,2,\cdots,5 \right)$

表 2. 均勻矩形平板在 SSSS 邊界下附帶一個彈簧-質量系統所得之最低五個特徵值(懸吊系統位於 $\xi_1 = x_1/a = 0.75$ ， $\eta_1 = y_1/b = 0.75$ and b/a=0.5）。

k_1^*	m_1^*	Methods	Eigenvalues (rad/sec)				
			ω_1	ω_2	ω_3	ω_4	ω_5
0.5	0.25	present	3.86702	95.41425	152.66339	248.07843	324.41109
		Ref.[5]	3.86505	95.43456	152.68843	248.08641	324.42236
1.0	0.5	present	3.86702	95.41356	152.66286	248.07797	324.41134
		Ref.[5]	3.86306	95.45412	152.71291	248.09395	324.43388

Note ： a=2.0m, b=1.0m, h=0.005m, $\upsilon = 0.3$ ，

$\rho = 7850 kg/m^3$ ， $m_p = \rho hab = 78.5 kg$ ，

$D_E = Eh^3/12(1-\upsilon^2) = 2.3478 \times 10^3 N-m$, $E = 2.051 \times 10^{11} N/m^2$

5.2. 攜帶任意個數懸吊系統的均勻矩形板

本節使用本文方法來計算一均勻矩形板攜帶任意個數懸吊系統的特徵值，以證明本文方法應用於一般受拘束均勻矩形板之適用性。

若有一均勻矩形板攜帶一個懸吊系統，其位置為 $\xi_1 = x_1/a = 0.75$, $\eta_1 = y_1/b = 0.75$ ，其彈簧常數為 $k_1 = k_p$ ，其懸吊質量為 $m_1 = 0.5m_p$ ，則均勻矩形板之前五個最低的特徵值 ω_i ($i = 1,2,\cdots,5$)(rad/sec)如表 3 所示。

表 3. 均勻矩形平板在四種邊界條件下附帶一個彈簧-質量系統所得之最低五個特徵值 (懸吊系統位於 $\xi_1 = x_1/a = 0.75$, $\eta_1 = y_1/b = 0.75$ and *b*/*a*=0.5)。

k_1^*	m_1^*	Boundary conditions	Eigenvalues (rad/sec)				
			ω_1	ω_2	ω_3	ω_4	ω_5
1.0	0.5	SSSS	3.86702	95.41356	152.66286	248.07797	324.41134
		SSSC	3.86702	134.04460	182.88172	271.08699	399.65117
		SCSC	3.86702	184.19115	223.90719	302.31799	423.38538
		SFSF	3.82903	53.21548	124.78718	203.96535	225.89612

Note ： *a*=2.0*m*, *b*=1.0*m*, *h*=0.005*m*, $\upsilon = 0.3$, $\rho = 7850\,kg/m^3$, $m_p = \rho hab = 78.5\,kg$, $D_E = Eh^3/12(1-\upsilon^2) = 2.3478\times10^3\,N-m$, $E = 2.051\times10^{11}\,N/m^2$

若有一均勻矩形板攜帶五個懸吊系統，該五個懸吊系統之位置、彈簧常數($k_i^*, i = 1,2,\cdots,5$)與懸吊質量($m_i^*, i = 1,2,\cdots,5$)如表 4 所示。則此一受拘束的均勻矩形板分別利用本文方法及依文獻[8]有限元素法(FEM) 撰寫程式所得到之前五個最低的特徵值 ω_i ($i = 1,2,\cdots,5$)(rad/sec)如表 5 所示。從表 5 中可以知道本文方法所得到之結果與有限元素法(FEM)所得結果均非常接近。

表 4. 均勻矩形平板附帶五個彈簧-質量懸吊系統其附帶位置與大小。

Locations of spring-mass $(\xi_i, \eta_i) = (x_i/a, y_i/b)$	(ξ_1, η_1)	(0.25,0.25)
	(ξ_2, η_2)	(0.25,0.75)
	(ξ_3, η_3)	(0.5,0.5)
	(ξ_4, η_4)	(0.75,0.25)
	(ξ_5, η_5)	(0.75,0.75)
Magnitudes of point masses ($m_v^* = m_v/m_p$)	m_1^*	0.5
	m_2^*	0.5
	m_3^*	0.5
	m_4^*	0.5
	m_5^*	0.5
Magnitudes of spring constants ($k_v^* = k_v/k_p$)	k_1^*	4.5
	k_2^*	4.5
	k_3^*	4.5
	k_4^*	4.5
	k_5^*	4.5

表 5. 均勻矩形平板在四種邊界條件下附帶五個彈簧-質量系統所得之最低五個特徵值(懸吊系統附帶位置與大小如表 4 所示)。

Boundary conditions	Methods	Eigenvalues (rad/sec)				
		ω_1	ω_2	ω_3	ω_4	ω_5
SSSS	Present	4.46700	4.63897	4.67763	4.68947	4.73608
	FEM	4.47776	4.65645	4.69378	4.69861	4.70083
SSSC	Present	4.50981	4.64012	4.67886	4.69016	4.73608
	FEM	4.51661	4.67061	4.69525	4.70157	4.70370
SCSC	Present	4.53458	4.56016	4.62442	4.64743	4.73608
	FEM	4.55605	4.68163	4.69850	4.70274	4.70511
SFSF	Present	4.31601	4.56016	4.61977	4.64739	4.73608
	FEM	4.31562	4.58080	4.68853	4.69038	4.69383

Note：a=2.0m, b=1.0m, h=0.005m, $\upsilon = 0.3$, $\rho = 7850\,kg/m^3$, $m_p = \rho hab = 78.5kg$, $E = 2.051\times10^{11}\,N/m^2$, and $D_E = Eh^3/12(1-\upsilon^2) = 2.3478\times10^3\,N-m$

陸、結論

本文所介紹的方法可適用於求解一均勻矩形板攜帶任意個數懸吊系統的特徵值，且具有合理之精確度。本文提出數值組合之技巧，可克服現有之閉式解析法因顯示式過於冗長而不適用於求解一均勻矩形板攜帶二個(含)以上之懸吊系統之自由振動分析的問題。

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